

Lecture XXXIV

Subspaces

In the previous lecture we have seen that there are two methods for finding the inverse of a square matrix A . In the first method, we use the fact that if A is non-singular, then the row-reduced form of $[A : I]$ is $[I : A^{-1}]$. In the second method, we take the matrix of cofactors A^* , consider its transpose $(A^*)^t$, and multiply it by $\frac{1}{\det A}$ to obtain the inverse of A : $\frac{1}{\det A}(A^*)^t$. The following theorem is an important determinant property.

Theorem 1 *Let A and B be two square matrices of the same size. Then*

$$|AB| = |A||B|.$$

Definition 1 *A set V is a vector subspace of \mathbf{R}^n if it satisfies the following properties:*

- (i) 0 is in V .
- (ii) For any \vec{x}, \vec{y} in V , $\vec{x} + \vec{y}$ is in V .
- (iii) For any \vec{x} in V and any scalar c in \mathbf{R} , $c\vec{x}$ is in V .

Clearly, the set containing only the origin is a vector subspace. In \mathbf{R}^3 , a vector subspace must be either the set previously mentioned, a line through the origin, a plane through the origin, or all of \mathbf{R}^3 .

Definition 2 *Let X be a vector in \mathbf{R}^n and let V be a vector subspace of \mathbf{R}^n . Let us denote by $\mathcal{A} = X + V$ the set of all the vectors $X + Y$, with Y in V . We say that \mathcal{A} is an affine subspace of \mathbf{R}^n .*

Let \mathcal{A} be a given affine subspace that is not formed by a single point only. Let X be a vector and let V be a vector space such that $\mathcal{A} = X + V$. Then the choice of V is unique, where X can be any vector in \mathcal{A} .

Definition 3 Let c_1, \dots, c_n be scalars and let Y_1, \dots, Y_n be contained in a vector subspace V . The vector $c_1Y_1 + c_2Y_2 + \dots + c_nY_n$ is called a linear combination of vectors Y_1, \dots, Y_n .

Theorem 2 Any linear combination of vectors from a vector subspace V is contained in V .

Definition 4 Let Y_1, \dots, Y_n be contained in a vector subspace V . We call the span of Y_1, \dots, Y_n and denote by $\text{span}(Y_1, \dots, Y_n)$ the set of all linear combinations of Y_1, \dots, Y_n .

For any vectors Y_1, \dots, Y_n in V , $\text{span}(Y_1, \dots, Y_n)$ is a vector subspace contained in V .

Definition 5 Vectors Y_1, \dots, Y_k are linearly independent if a linear combination $c_1Y_1 + \dots + c_kY_k$ is equal to 0 if and only if $c_1 = \dots = c_k = 0$.

Definition 6 If $Y_1, \dots, Y_k \in V$ are linearly independent and the span of Y_1, \dots, Y_k is V , then the set Y_1, \dots, Y_k is called a basis for V . Any point in V can be expressed as a linear combination of Y_1, \dots, Y_k in a unique way.