

Lecture XXXIII

Determinants; Matrix Algebra

1 Determinants

For a square matrix A , the determinant of A has the following properties:

1. Interchanging two rows of the matrix multiplies the value of the determinant by -1 .
2. If there exists two identical rows, then the value of the determinant is 0 .
3. If we multiply a row by a scalar c , then the value of the determinant is also multiplied by c .
4. Applying the β operation to the matrix leaves the value of the determinant unchanged.
5. If the determinant is 0 , then there exists a row that can be written as a linear combination of the other rows.
6. Properties (1) through (5) also hold for columns instead of rows.

In addition to the Laplace expansion method of finding determinants, there is a second method that is faster in most cases. This is a simplified *row-reduction* that brings the matrix in to a form called *row-echelon* matrix. In this method we use operations β and γ . The row-echelon matrix is different from the row-reduced matrix in that:

1. pivots can have other values than 1 , since we do not use α operations;
2. in the column of a pivot, we only need to get 0 's below the pivot.

If each column has a pivot, then the determinant of the initial matrix A is equal to the product of the pivots times $(-1)^k$, where k is the number of γ operations we applied to obtain the row-echelon form. If a column has no pivots, then there will be a row containing only 0's, so the determinant will be 0. Let us define the *rank* of A as the number of pivots in the row-echelon form. The following sentences hold:

1. If $|A| = 0$, then $\text{rank}(A) < n$. In this case we say that A is a *singular* matrix.
2. If $|A| \neq 0$, then $\text{rank}(A) = n$. In this case we say that A is a *non-singular* matrix.

2 Matrix Algebra

We saw in the previous lecture that we can multiply matrices with column vectors. Let us now define in a larger sense matrix multiplication. In order to define the product AB of matrices A and B , A must have dimensions $m \times n$ and B must have dimensions $n \times p$. In other words, the number of columns in A must equal the number of rows in B . Then multiplying A with the i -th column of B , we obtain a column m -vector. This column-vector will be the i -th column of AB . Hence AB is an $m \times p$ matrix. It can be easily observed that BA exists as well only if $m = p$. In this case AB is an $m \times m$ matrix and BA is an $n \times n$ matrix. So even if AB and BA both exist, they may not have the same size. Furthermore, even if AB and BA have the same size, it is possible that $AB \neq BA$. Hence the multiplication of matrices is non-commutative. Let A be an $m \times n$ matrix, let B be an $n \times p$ matrix, and let C be a $p \times l$ matrix. Then $(AB)C$ and $A(BC)$ both exist and furthermore

$$(AB)C = A(BC)$$

Hence, the multiplication of matrices is associative. The following properties also hold:

1. $AI_n = A$ and $I_n A = A$, if the products exist, where I_n is the identity $n \times n$ matrix we defined in the previous lecture.

2. $A0 = 0$ and $0A = 0$, if the products exist, where 0 a matrix with all its entries equal to 0 .
3. $A(B + C) = AB + AC$
4. $(A + B)C = AC + BC$

For square matrices A, B , if $AB = I_n$, we say that A is *inverse to* B . The following properties hold:

1. If $AB = I$, then $BA = I$, i.e. if A is inverse to B , then B is inverse to A .
2. If $BA = I$ and $CA = I$, then $B = C$.

Hence the inverse of a square matrix A is unique, if it exists. We denote it by A^{-1} .

Theorem 1 *The inverse of matrix A exists if and only if A is non-singular.*

We can use the inverse to solve systems of equations. Consider the system $AX = D$, where A is a non-singular square matrix. Then $X = A^{-1}D$. One method to find the inverse of a matrix A is the following: consider the matrix of cofactors of A , transpose it and multiply it by $\frac{1}{|A|}$. The result is the inverse of A . A second method involves the $n \times 2n$ matrix $[A : I_n]$. If we apply row-reduction to this matrix, we get a matrix $[B : C]$, with B and C $n \times n$ matrices. If $B \neq I$, then A is singular, so A^{-1} does not exist. If $B = I$, then A is non-singular and $C = A^{-1}$.