Lecture II
Vectors and Vector Algebra

A set S of rays is called a *direction* if it satisfies the following laws:

(1) Any two rays in S have the same direction.

(2) Every ray that has the same direction as some member of S is in S.

A *vector* $\vec{A}$ consists of a non-negative real number, called the *magnitude* of the vector, and a direction. We denote the magnitude of $\vec{A}$ by $|\vec{A}|$.
A vector $\vec{A}$ such that $|\vec{A}| = 1$ is called a *unit vector*.

We will now describe four basic algebraic operations with vectors in $\mathbf{E^3}$:

1 Multiplication by a scalar

Let $\vec{A}$ by a vector, and let c by a real number. Multpying $\vec{A}$ by the scalar c, we obtain a vector denoted by $c\vec{A}$. The magnitude of the result is given by $|c\vec{A}| = |c||\vec{A}|$. The direction of $c\vec{A}$ is the same as the direction of $\vec{A}$ if $c \geq 0$, and opposite to the direction of $\vec{A}$ if $c < 0$.

2 Addition of vectors

The sum of two vectors $\vec{A}$ and $\vec{B}$ is denoted simply by $\vec{A} + \vec{B}$. We can define this sum geometrically. Translate $\vec{B}$ such that its start point is the end-point of $\vec{A}$. Then $\vec{A}+\vec{B}$ will be the vector having the same start-point as $\vec{A}$ and the same end-point as $\vec{B}$.

3 Scalar product (Dot product)

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is a scalar quantity denoted by $\vec{A} \cdot \vec{B}$. Its value is $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|cos\theta$, where θ is the angle made by $\vec{A}$ and $\vec{B}$ if we translate $\vec{B}$ such that it has the same start-point as $\vec{A}$. We observe that $|\vec{B}|cos\theta$ is the length of the projection of $\vec{B}$ on $\vec{A}$.

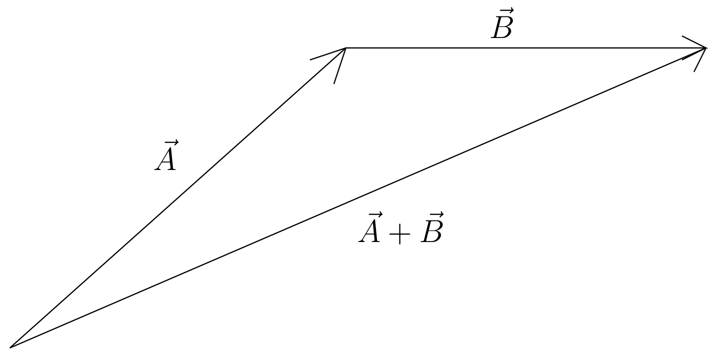

Figure 1: Vector addition

The magnitude of $\vec{A}$ is equal to the square root of the dot product of A with itself:

$$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{B}}$$

Hence $\frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{\sqrt{\vec{A}\cdot\vec{A}}}$ is a unit vector with the same direction as A.

4 Vector product (Cross product)

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector denoted by $\vec{A}\times\vec{B}$. The magnitude of the cross product is given by:

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|sin\theta.$$

Let $\vec{A}$ and $\vec{B}$ have the same start-point P and end-points Q_1 and Q_2, respectively. Let M be the plane of $\vec{A}$ and $\vec{B}$. The direction of $\vec{A}\times\vec{B}$ is normal to M in the manner established by the right-hand rule: if a right hand is placed at P and the fingers are curling from PQ_1 to TQ_2 through the angle smaller than π, then the thumb indicates the direction of $\vec{A}\times\vec{B}$.

The following are some properties of these basic vector operations.

1. $(a+b)\vec{C} = a\vec{C} + b\vec{C}$
2. $(ab)\vec{C} = a(b\vec{C})$
3. $a(\vec{B}+\vec{C}) = a\vec{B} + a\vec{C}$

4. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

5. $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

6. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$