Lecture XIV

Multiple Integrals

1 Integrals of one-variable functions

For a real-valued function f defined on an interval $[a, b]$, the *integral of f over $[a, b]$*, denoted by $\int_a^b f dx$, is defined as follows:

$$\int_a^b f dx = \lim_{n\to\infty, \max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i,$$

where $a = x_1 < \ldots < x_i < x_{i+1} < \cdots < x_n = b$, $\Delta x_i = x_i - x_{i-1}$ and $x_i^* \in [x_{i-1}, x_i]$ for all $i \geq 2, i \leq n$. We will now give a few properties and theorems about integrals in one-variable calculus.

Theorem 1 (First Existence Theorem) *Let f be a one-variable function. If f is continuous on $[a, b]$, then $\int_a^b f dx$ exists.*

Let f and g be two functions defined on $[a, b]$ such that $\int_a^b f dx$ and $\int_a^b g dx$ exist. Then the following properties hold:

1. Let c be such that $a < c < b$. Then the following integrals exist and the equality holds:
$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx.$$
2. The integral in the left side of the equality exists and the equality holds:
$$\int_a^b (f+g) dx = \int_a^b f dx + \int_a^b g dx.$$
3. If c is a constant, the integral in the left side of the equality exists and the equality holds:
$$\int_a^b (cf) dx = c\int_a^b f dx.$$

4. If m_1 and m_2 are two constants such that $m_1 \leq f(x) \leq m_2$ for all $x \in [a, b]$, then

$$m_1(b-a) \leq \int_a^b f dx \leq m_2(b-a).$$

2 Scalar-valued integrals in $\mathbf{E}^2$ and $\mathbf{E}^3$

If f is a scalar-valued function defined on $\mathbf{E^2}$ or $\mathbf{E^3}$, how do we define the integrals of f on a region R? We can only do this on a special kind of region, called *regular*, which we will define in section 5.

Definition 1 *Let R be a regular region in $\mathbf{E^2}$ and f a scalar field on R. The* integral of f on R, *denoted by $\int\int_R f dA$, is defined as follows:*

$$\int\int_R f dA = \lim_{\substack{n\to\infty \\ \max d_i \to 0}} \sum_{i=1}^{n} f(P_i^*)\Delta A_i,$$

where $A_1, \ldots, A_n$ form a subdivision of R into elementary *regions, ΔA_i is the area of A_i, P_i^* is a point in A_i and d_i is the* diameter *of A_i, i.e. the longest distance between two points in A_i, for all $i \geq 1, i \leq n$.*

Theorem 2 (Second Existence Theorem) *If R is a regular region in $\mathbf{E^2}$ and f is continuous on R, then $\int\int_R f dA$ exists.*

For $\mathbf{E^3}$, the definition of the integral is very similar.

Definition 2 *Let R be a regular region in $\mathbf{E^3}$ and f a scalar field on R. The* integral of f on R, *denoted by $\int\int_R f dV$, is defined as follows:*

$$\int\int\int_R f dV = \lim_{\substack{n\to\infty \\ \max d_i \to 0}} \sum_{i=1}^{n} f(P_i^*)\Delta V_i,$$

where $V_1, \ldots, V_n$ form a subdivision of R into elementary *regions, ΔV_i is the volume of V_i, P_i^* is a point in V_i and d_i is the* diameter *of V_i, for all $i \geq 1, i \leq n$.*

Theorem 3 (Third Existence Theorem) *If R is a regular region in $\mathbf{E^3}$ and f is continuous on R, then $\int\int\int_R f dV$ exists.*

Remark: If we consider a constant function $f(x, y) = 1$ over a region R, then $\int\int_R f dA = \int\int_R dA$ is equal to the area of R.

Sometimes it is useful not to use the Cartesian coordinate system. For example, let R be the disc bounded by the curve $x^2 + y^2 = 4$. Let $f(x, y) = x^2 + y^2$. We want to find the integral $\int\int_R f(x, y)dA$. We divide this disc into n concentric rings, and further divide these rings by radial segments. For each ring j , we choose a r_j^* such that the area of the ring is $2\pi r_j^* \Delta r_j$, where Δr_j is the difference between the radii of rings j and $j - 1$. Then we can choose a point P^* in each subregion of the ring j such that $f(P^*) = (r_j^*)^2$. Hence the Reimann sum for this choice of subdivision is $\sum_{j=1}^m 2\pi(r_j^*)^3\Delta r_j$. Now we can view this sum as the Reimann sum of a function of r. Hence

$$\int\int_R x^2 + y^2 dA = \int_0^2 2\pi r^3 dr = 8\pi.$$

3 Properties of multiple integrals

1. Let R be a regular region in $\mathbf{E^3}$ and let R_1 and R_2 be two regular disjoint regions such that $R = R_1 \cup R_2$. If $\int\int\int_R f dV$ exists, then the other integrals in the following equality exists, and the equality holds:

$$\int\int\int_R f dV = \int\int\int_{R_1} f dV + \int\int\int_{R_2} f dV.$$

   The equivalent for $\mathbf{E^2}$ holds as well.

   Here is an application of this property. Let $f(x, y) = x$ for all x, y, and let R be the disc bounded by $x^2 + y^2 = 1$. Let R_1 be the part of R that is left of Oy, and let R_2 be the part of R that is right of Oy. Then $\int\int_R f DA = \int\int_{R_1} f dA + \int\int_{R_2} f dA$. But by symmetry, $\int\int_{R_1} x dA = \int\int_{R_2} -x dA$, so $\int\int_R f DA = 0$.

2. Let R be a regular region in $\mathbf{E^3}$ and f a function on R such that $\int\int\int_R f dV$ exists. If m_1 and m_2 are two constants such that $m_1 \leq f(P) \leq m_2$ for all points P in R, then the following inequalities hold:

$$m_1 V \leq \int\int\int_R f dV =\leq m_2 V,$$

   where V is the volume of R. The equivalent for $\mathbf{E^2}$ also holds.

   As an application of this property, consider the region R to be the square having $(0, 0)$ and $(2, 2)$ opposite vertices. Let $f(x, y) = 2 + \frac{xy}{100}$ for all

$(x, y) \in R$. Then we can aproximate f by $2 \leq f(x, y) \leq 2.04$ for all $(x, y) \in R$. Hence by the property above, since the area of R is equal to 4, we have that $8 \leq \int\int_R f dA \leq 8.16$.

3. Let R be a regular region in $\mathbf{E^3}$ and let f, g be two functions on R such that $\int\int\int_R f dV$ and $\int\int\int_R g dV$ exist. Then the integral on the left side of the following equality exists and the equality holds:

$$\int\int\int_R (f+g) dV = \int\int\int_R f dV + \int\int\int_R g dV.$$

   The equivalent property for $\mathbf{E^2}$ holds as well.

   Let us consider R to be the disc bounded by $x^2 + y^2 = 4$. Let $f(x, y) = x^2$ and $g(x, y) = y^2$ be two functions defined on R. By symmetry, $\int\int_R f dA = \int\int_R g dA$, so applying the property above, we get that

$$\int\int_R x^2 dA = \frac{1}{2}\int\int_R (x^2 + y^2) dA.$$

   As we've seen in the example at the end of section 2, $\int\int_R (x^2+y^2) dA = 8\pi$, hence

$$\int\int x^2 dA = 4\pi.$$

4. Let R be a regular region in $\mathbf{E^3}$ and f a function on R such that $\int\int\int_R f dV$ exists. If c is a constant, the integral in the left side of the equality exists and the equality holds:

$$\int\int\int_R (cf) dV = c\int\int\int_R f dV.$$

4 Vector-valued integrals

Let $\vec{F}(x, y, z)$ be a vector-valued function defined on $\mathbf{E^3}$, $\vec{F}(x, y, z) = L(x, y, z)\hat{\imath} + M(x, y, z)\hat{\jmath} + N(x, y, z)\hat{k}$. The integral of $\vec{F}$ exists if the integrals of L, M, and N exist, and then the following equality holds:

$$\int\int\int_R \vec{F}(x, y, z) dV =$$

$$\left(\int\int\int L dV\right)\hat{\imath} + \left(\int\int\int M dV\right)\hat{\jmath} + \left(\int\int\int N dV\right)\hat{k}.$$

As an application, consider an object that ocupies a region R in space and has density $\delta(x,y,z)$ at any point $P(x,y,z)$. The center of mass of this object s given by

$$\vec{C}_M = \frac{1}{M}\int\int\int_R \delta(x,y,z)\vec{OP}(x,y,z)dV,$$

where M is the total mass of the object, i. e. $M = \int\int\int_R \delta(x,y,z)dV$. Hence

$$C_M = \frac{1}{M}\int\int\int_R \delta(x,y,z)(x\hat{\imath}+y\hat{\jmath}+z\hat{k})dV =$$

$$= \frac{1}{M}\int\int\int_R x\delta(x,y,z)dV\hat{\imath}+$$

$$+\frac{1}{M}\int\int\int_R y\delta(x,y,z)dV\hat{\jmath} + \frac{1}{M}\int\int\int_R z\delta(x,y,z)dV\hat{k}.$$

5 Further remarks

In the existence theorems, the condition that f be continuous can be replaced with *piecewise* continuous.

Definition 3 *A function f is said to be* piecewise continuous *on a regular region R if it is defined on R and it is possible to divide R in a finite number of regular subregions $R_1,\ldots,R_n$ such that for each subregion R_i, the restiction of f to the interior of R_i can be extended to a function f_i defined on the boundary of R_i as well, such that f_i is continuous. It is not necessary that f_i coincide with f on the boundary of R_i.*

Now let us define the concept of *regular region*. For this we will need to define first what an *elementary region* is.

Definition 4 *A region R is called* elementary *if there is a piecewise smooth, simple, closed curve such that R is the union of the curve and its interior.*

Definition 5 *A region R is called* regular *if it can be divided into a finite number of elementary regions in the following fashion: the regions form a connected whole and neighboring regions touch along a unique segment of common boundary curve, that has positive length.*

Let R be the region in $\mathbf{E^2}$ defined by all points (x, y) such that $x^2 + y^2 \leq 1$ and both x and y are irrational numbers. Then R is an example of a region that is not regular.