

# Lecture XXXII

## Row Reduction; Determinants

### 1 Row Reduction

Recall the 3 elementary operations we will use to solve the system of equations  $AX = D$ :

- ( $\alpha$ ) multiplying an equation by a non-zero scalar;
- ( $\beta$ ) adding to an equation some multiple of a different equation;
- ( $\gamma$ ) interchanging two equations.

We call the matrix  $[A : D]$  the *augmented matrix* of the system  $AX = D$ . We will use  $\alpha, \beta$ , and  $\gamma$  to bring  $B = [A : D]$ , or, in the case of homogeneous systems,  $B = A$  to a *row-reduced form*.

**Definition 1** *The row-reduced form of a matrix  $B$  is a matrix obtained from  $B$  by applying the elementary operations that has the following properties:*

1. *Every non-zero row has 1 as its first non-zero element from left to right. This element is called the pivot of the row.*
2. *Each pivot occurs in a column to the right of all pivots from upper rows.*
3. *All zero rows are at the bottom of the matrix.*
4. *For any pivot, all other elements in its column are equal to zero.*

We obtain this form through the method of *row-reduction*, which is described below:

1. Choose the first non-zero column from left to right, then choose a row having a non-zero element on this column.

2. Move this row on top of the matrix.
3. Multiply this row by  $\frac{1}{a}$ , where  $a$  is its first non-zero element from left to right.
4. For any row having  $b \neq 0$  on the column on which the chosen row has its first non-zero element, add  $-b$  times the chosen row to it.
5. Repeat the above steps, moving the selected row to the second position, then the third position, and so on.

If, in the case of a nonhomogeneous system, the last column has a pivot, then the system doesn't have solutions. If the last column doesn't contain any pivot and there are other columns as well that don't contain pivots, then the system has infinitely many solutions. Suppose a nonhomogeneous system has the row-reduced augmented matrix

$$\begin{bmatrix} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then we can write the equations of the system:

$$\begin{aligned} x_1 - 7x_3 &= 0, & \text{hence} & & x_1 &= 7x_3 \\ x_2 + 3x_3 &= 2, & \text{hence} & & x_2 &= 2 - 3x_3 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$$

Hence  $\vec{X} = \vec{A} + t\vec{B}$ , where

$$\vec{A} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{B} = \begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$$

In this case the solution set forms a line in 3-dimensional space. Generally, for systems with 3 unknowns, the solution set is given by  $\vec{X} = \vec{C} + t_1\vec{B}_1 + t_2\vec{B}_2$ , where  $\vec{C}, \vec{B}_1, \vec{B}_2$  are 3-vectors, possibly equal to  $\vec{0}$ . For homogeneous systems,  $\vec{C} = \vec{0}$ .

## 2 Determinants

Consider a square  $n \times n$  matrix  $A$ . The *determinant* of the matrix is a scalar depending on  $A$ , denoted by  $|A|$ . For a  $1 \times 1$  matrix, the determinant of the matrix is simply the value of its element. Using the notion of *minors*, we can define a recurrent formula for determinants.

**Definition 2** *The minor of an element  $a_{ij}$  of a square matrix is the determinant of the matrix formed from the initial matrix excluding the row  $i$  and the column  $j$  containing this element.*

We denote by  $m_{ij}$  the minor of  $a_{ij}$ . Consider the  $n \times n$  matrix having as elements the signs  $+$  and  $-$ , in a checkerboard pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

We call the *cofactor* of an element  $a_{ij}$ , the minor of  $a_{ij}$  multiplied by  $+1$  or  $-1$ , depending on the position of the element in the above  $\pm$  matrix. The cofactor of  $a_{ij}$  has the value  $(-1)^{i+j}m_{ij}$ . For any column or row of  $A$ , the determinant of  $A$  is given by the formula

$$|A| = \sum (-1)^{i+j} a_{ij} m_{ij},$$

where the sum is on the column or row, i.e.  $j$  is constant and  $i$  goes from 1 to  $m$ , or  $i$  is constant and  $j$  goes from 1 to  $n$ .