Lecture IX

Linear Approximation

1 One-variable Functions

Let f be a one-variable function on a domain D. If f is differentiable at $x = c$, we say f has a linear approximation, which we define in the following way.

Definition 1 *If f is a function differentiable at c, for any x define $\Delta f = f(x) - f(c)$ and $\Delta x = x - c$. There exist a scalar A_c and a function $\epsilon_c(x)$ such that*

$$\Delta f = A_c \Delta x + \epsilon_c(x)\Delta x \quad \textit{and} \quad \lim_{x \to c} \epsilon_c(x) = 0.$$

Then

$$A_c = f'(c) \quad \textit{and} \quad \epsilon_c(x) = \frac{\Delta f}{\Delta x} - f'(c).$$

We say that f has a linear approximation at c *and that the expression $\Delta f_{app} = f'(c)\Delta x$ is a* linear approximation formula for f at c. *The function ϵ_c is called the* relative error function for f at c.

2 Multivariable Functions

Definition 2 *Let f be a multilinear function and let P be a point in its domain D in $\mathbf{E^2}$. Define $\Delta f = f(a + \Delta x, b + \Delta y) - f(a, b)$. We say that f has* linear approximation at P *if there exist scalars A, B and functions ϵ_1, ϵ_2 such that:*

1. *for all $Q \in D$,$\Delta f = A\Delta x + B\Delta y + \epsilon_1(Q)\Delta x + \epsilon_2(Q)\Delta y$.*

2. $\lim_{Q \to P} \epsilon_1(Q) = 0$ *and* $\lim_{Q \to P} \epsilon_2(Q) = 0$.

Theorem 1 *If f has a linear approximation at P, then f has partial derivatives at P and $\frac{\partial f}{\partial x}|_P = A$, $\frac{\partial f}{\partial y}|_P = B$.*

Theorem 2 *If f has continuous partial derivatives on all the domain, then f has a linear approximation.*

For a multivariable function f that has a linear approximation at P, we define $\Delta f_{app} = \frac{\partial f}{\partial x}|_P \Delta x + \frac{\partial f}{\partial y}|_P \Delta y$, the *linear approximation formula for f at P.*

For a three-variable function f, we say that f has a linear approximation at P if there exist A, B, C scalars and $\epsilon_1, \epsilon_2, \epsilon_3$ functions such that $\Delta f = A\Delta x + B\Delta y + C\Delta z + \epsilon_1(Q)\Delta x + \epsilon_2(Q)\Delta y + \epsilon_3(Q)\Delta z$ and $\lim_{Q\to P} \epsilon_1(Q) = 0$, $\lim_{Q\to P} \epsilon_2(Q) = 0$, $\lim_{Q\to P} \epsilon_3(Q) = 0$. If f has a linear approximation at P, then f has partial derivatives at P, and $A = \frac{\partial f}{\partial x}|_P$, $B = \frac{\partial f}{\partial y}|_P$, $C = \frac{\partial f}{\partial z}|_P$. The linear approximation formula for f at P is $\Delta f_{app} = \frac{\partial f}{\partial x}|_P \Delta x + \frac{\partial f}{\partial y}|_P \Delta y + \frac{\partial f}{\partial z}|_P \Delta z$.