Lecture VII
Paths and Curves

First we go through several basic notions about paths. Let $\vec{R}(t)$ on $[a, b]$ be a given path.

Definition 1 *$\vec{R}(t)$ is called* elementary *if for every pair (t_1, t_2), with t_1 and t_2 distinct in $[a, b]$, $\vec{R}(t_1) \neq \vec{R}(t_2)$.*

Definition 2 *$\vec{R}(t)$ is called* simple *if for every pair (t_1, t_2), with t_1 and t_2 distinct in $[a, b]$, except possibly for the pair (a, b), $\vec{R}(t_1) \neq \vec{R}(t_2)$.*

Definition 3 *$\vec{R}(t)$ is called* closed *if $\vec{R}(a) = \vec{R}(b)$.*

A closed and simple path is called a *loop.* A loop in $\mathbf{E^2}$ is called a *Jordan curve.* Given a Jordan curve $\mathcal{C}$, $\mathbf{E^2}$ can be divided into three regions, two bounded and one unbounded. The bounded regions are $\mathcal{C}$ and R_i, the interior of $\mathcal{C}$. The unbounded region is R_e, the exterior of $\mathcal{C}$.

Definition 4 *A* directed curve *is a curve along which we have specified a direction.*

Given a curve C, finding a path for C lets us apply calculus techniques. If C is the circle of radius R and center O, then a path for C is $\vec{R}(t) = R\cos t\hat{\imath} + R\sin t\hat{\jmath}$. Let C be the curve generated by a point P fixed on a circle of radius a rolling without slipping along the x axis. Suppose the curve begins with P at O and ends when P again touches the x axis. The circle makes a full rotation, so it would be advisable to take θ, the angle through which the circle rotates, as the variable for a path for C. The coordinates of the center of the circle are $(a\theta, a)$, so the coordinates of P are $(a(\theta - \sin\theta), a(1 - \cos\theta))$. Hence the path is $\vec{R}(\theta) = a(\theta - \sin\theta)\hat{\imath} + a(1 - \cos\theta)\hat{\jmath}$, with θ from 0 to 2π.

Let $\vec{R}$ be a path for the curve C and let P be the point with position vector $\vec{R}(t_0)$. Recall that $\lim_{\Delta t \to 0} \frac{\vec{R}(t_0 + \Delta t) - \vec{R}(t_0)}{\Delta t} = \frac{d\vec{R}}{dt}|_{t_0} = \vec{v}(P)$ is a tangent vector to C at P.

Definition 5 *Let P be a point with position vector $\vec{R}(t_0)$. If $\frac{d\vec{R}}{dt}|_{t_0}$ exists and is not equal to 0, then we denote by $\hat{T}_P$ the* unit tangent vector at P, *the vector*

$$\hat{T}_P = \frac{d\vec{R}}{dt}\Big|_{t_0} \Bigg/ \left|\frac{d\vec{R}}{dt}\Big|_{t_0}\right|$$

Let s be the arc length of C, $s(t_0) = \int_a^{t_0} |\frac{d\vec{R}}{dt}|dt$. Then

$$\vec{v} = \frac{d\vec{R}}{dt} = \frac{d\vec{R}}{ds} \cdot \frac{ds}{dt} \quad \textit{and} \quad \hat{T} = \frac{d\vec{R}}{ds}, \quad \textit{so}$$

$$\vec{v} = \hat{T}\frac{ds}{dt} = \hat{T} \cdot \dot{s}$$

Definition 6 *For a point P on C, $|\frac{d\hat{T}}{ds}|$ at P is called the* curvature of C at P *and is denoted by k.*

Definition 7 *If $\frac{d\hat{T}}{ds} \neq 0$ at P, then we can define the* unit normal vector to C at P, $\hat{N}_P = \frac{d\hat{T}}{ds} \Big/ |\frac{d\hat{T}}{ds}|$.

Clearly, $k\hat{N} = \frac{d\hat{T}}{ds}$ and $\hat{N}_P$ is orthogonal to $\hat{T}_P$. Since $\vec{v} = \dot{s}\hat{T}$, we get that $\vec{a} = \frac{d(\dot{s}\hat{T})}{dt} = \ddot{s}\hat{T} + \dot{s}\dot{\hat{T}}$. Since $\dot{\hat{T}} = \frac{d\hat{T}}{dt} = \frac{d\hat{T}}{ds}\frac{ds}{dt} = k\hat{N}\dot{s}$, we have that

$$\vec{a} = \ddot{s}\hat{T} + \dot{s}\dot{\hat{T}} = \ddot{s}\hat{T} + k\dot{s}^2\hat{N}$$

Definition 8 *Let P be a point where $\hat{T}_P$ and $\hat{N}_P$ both exist. The vector $\hat{B}_P = \hat{T}_P \times \hat{N}_P$ is called the* unit binormal vector at P.

The vectors $\hat{T}_P, \hat{N}_P$, and $\hat{B}_P$ form a frame at P, that is called the *moving frame at P*.