Lecture V
Calculus of One-Variable Functions

Let us first review some definitions in calculus on real numbers.

In order to define limit on the real numbers we will use the concept of *funnel functions*.

Definition 1 *A function* $\epsilon(t)$ *on* $[0, d]$ *is called a* funnel function *if it has the following properties:*

1. *It is strictly increasing.*
2. *For every* $a > 0$*, there exists* t*,* $0 < t \leq d$*, such that* $0 < \epsilon(t) < a$*.*

Definition 2 *Let* $f(x)$ *be a scalar function and let* l *be a real number. We say that* the limit of f at c is l *and we denote this by*

$$\lim_{x \to c} f(x) = l$$

if for some $d > 0$ *there exists a funnel function* $\epsilon(t)$ *on* $[0, d]$ *such that for every* x *with* $0 < |x - c| \leq d$ *it follows that* $|f(x) - l| \leq |\epsilon(|x - c|)$*.*

For a real function f and c, h real numbers, the fraction $\frac{f(c+h)-f(c)}{h}$ is called *difference quotient* and is denoted by $\frac{\Delta f}{\Delta x}$.

Definition 3 *For a real function* f*, if* $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ *exists and has a real value, we call this value the* derivative of f at c *and denote it by* $f'(c)$ *or* $\frac{df}{dt}|_c$*. We say that* f *is* differentiable at c*.*

Definition 4 *A real function* f *is* continuous at c *if* $\lim_{x \to c} f(x) = f(c)$*.*

We say that f is continuous on $[a, b]$ if f is continuous at all $c \in [a, b]$. We will now see how the limit and derivative concepts act on the sum and product of functions.

Let f and g be two real functions such that $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$. Then :

1. $\lim_{x\to c}(f+g)(x) = l_1 + l_2$.

2. $\lim_{x\to c}(fg)(x) = l_1 l_2$.

Let f and g be two real functions differentiable at c. The two properties below mean that the derivates exists and are equal with the given values.

1. $(f+g)'(c) = f'(c) + g'(c)$.

2. $(fg)'(c) = f(c)g'(c) + g(c)f'(c)$.

Definition 5 *Let $f(t)$ and $F(t)$ be two real function. The function $F(t)$ is called the* indefinite integral *of $f(t)$ if $F(t)$ is differentiable and $F'(t) = f(t)$ for all t. We denote this by $F(t) = \int f(t)dt + \mathscr{C}$. This is due to the fact that the indefinite integral of a function f is not unique, and the difference between two indefinite integrals of the same function is always a constant function.*

Let us now explain what the *definite integral* of f on $[a,b]$ is. Take $\delta > 0$ and divide $[a,b]$ in subintervals $[x_0,x_1],[x_1,x_2],\ldots,[x_{n-1},x_n]$, smaller than δ, where $x_0 = a$ and $x_n = b$. Denote $\Delta x_i = x_i - x_{i-1}$ for all $0 < i \leq n$. Then $f(x_1)\Delta x_1 + \ldots + f(x_n)\Delta x_n$ is called a *Riemann sum* of *mesh* δ on $[a,b]$ for the functon f. If the limit of the Riemann sums exists as $\delta \to 0$, then this limit is called the *definite integral of f on* $[a,b]$ and it is denoted by $\int_a^b f(t)dt$.

We will now review how the definitions above extend to one-variable vector functions.

Let $\vec{A}(t)$ be a vector function on an interval $[a,b]$.

Definition 6 *For a vector $\vec{L}$, if* $\lim_{t\to c}|\vec{A}(t) - \vec{L}| = 0$, *we say that the* limit of $\vec{A}(t)$ as t goes to c *is $\vec{L}$ and denote this by* $\lim_{t\to c}\vec{A}(t) = \vec{L}$.

Here are some basic facts about vector limit. Let $\vec{A}(t)$ and $\vec{B}(t)$ be two vector functions such that $\lim_{t\to c}\vec{A}(t) = \vec{L_1}$ and $\lim_{t\to c}\vec{B}(t) = \vec{L_2}$. Then:

1. $\lim_{t\to c}(\vec{A}(t) + \vec{B}(t)) = \vec{L_1} + \vec{L_2}$.

2. If $a(t)$ is a scalar function such that $lim_{t\to c}a(t) = l$, then $\lim_{t\to c}(a(t)\vec{A}(t)) = l\vec{L_1}$.

3. $\lim_{t\to c}(\vec{A}(t)\cdot\vec{B}(t)) = \vec{L_1}\cdot\vec{L_2}$.

4. $\lim_{t\to c}(\vec{A}(t)\times\vec{B}(t)) = \vec{L_1}\times\vec{L_2}$.

Definition 7 *A vector function* $\vec{A}(t)$ *is* continuous at c *if* $\lim_{t\to c}\vec{A}(t) = \vec{A}(c)$.

A vector function $\vec{A}(t)$ is said to be continuous on $[a,b]$ if it is continuous at all $t\in[a,b]$. If $\vec{A}(t)$ and $\vec{B}(t)$ are continuous functions, then $\vec{A}(t)\cdot\vec{B}(t)$ is a continuous scalar function, and $\vec{A}(t)+\vec{B}(t)$ and $\vec{A}(t)\times\vec{B}(t)$ are continuous vector functions. Also, if $a(t)$ is a continuous scalar function, $a(t)\vec{A}(t)$ is a continuous vector function.

Definition 8 *Let* $\frac{\Delta\vec{A}}{\Delta t}$ *be the difference quotient of a vector function* $\vec{A}(t)$. *If there exists* $\vec{D}$ *such that* $\lim_{t\to c}\frac{\Delta\vec{A}}{\Delta t} = \vec{D}$, *then* $\vec{D}$ *is called the* derivative of $\vec{A}(t)$ at c. *This is denoted by* $\vec{D} = \frac{d\vec{A}}{dt}|_c$. *We say that* $\vec{A}(t)$ *is* differentiable at c.

A function $\vec{A}(t)$ is called *differentiable on* $[a,b]$ if it is differentiable at every c on $[a,b]$. Let $\vec{A}(t) = a_1(t)\hat{\imath} + a_2(t)\hat{\jmath} + a_3(t)\hat{k}$ be a vector function. Then $a_1(t), a_2(t)$, and $a_3(t)$ are differentiable on $[a,b]$ if and only if $\vec{A}(t)$ is differentiable on $[a,b]$. In this case,

$$\frac{d\vec{A}}{dt} = \frac{da_1t}{dt}\hat{\imath} + \frac{da_2t}{dt}\hat{\jmath} + \frac{da_3t}{dt}\hat{k}.$$

Indefinite and definite integrals for one-variable vector functions are defined in the exact same manner as for scalar functions.