Problem Set VII Solutions

1. $\oint$ 16.4 2.

   (a) The expression for the moment of inertia of the solid R is

   $$\int\int\int_R \delta(x^2+y^2)dV = M\frac{\int\int\int_R (x^2+y^2)dV}{\int\int\int_R dV}.$$

   (b) We use $r^2 = x^2+y^2$ and we multiply the function that we integrate by r, since we switch to cylindrical coordinates:

   $$M\frac{\int\int\int_R (x^2+y^2)dV}{\int\int\int_R dV} = M\frac{\int_0^2\int_0^{2\pi}\int_z^2 r^3 drd\theta dz}{\int_0^2\int_0^{2\pi}\int_z^2 rdrd\theta dz}$$

   (c)

   $$M\frac{\int\int\int_R (x^2+y^2)dV}{\int\int\int_R dV} = M\frac{\int_0^2\int_0^{2\pi}\int_z^2 r^3 drd\theta dz}{\int_0^2\int_0^{2\pi}\int_z^2 rdrd\theta dz} =$$

   $$= \frac{M}{2}\frac{\int_0^2\int_0^{2\pi}(16-z^4)d\theta dz}{\int_0^2\int_0^{2\pi}(4-z^2)d\theta dz} = \frac{M}{2}\frac{\int_0^2(16-z^4)dz}{\int_0^2(4-z^2)dz} =$$

   $$= \frac{M}{2}\frac{32\frac{4}{5}}{8\frac{2}{3}} = \frac{12}{5}M.$$

2. $\oint$16.4 4.

   Clearly r goes from 0 to 2, θ goes from 0 to π , since for $\theta > \pi$, y is negative, so $z = y$ is negative. Also, z goes from 0 to $y = r\sin\theta$. Hence the mass is:

   $$\int_0^2\int_0^{\pi}\int_0^{r\sin\theta}\delta(r,\theta,z)rdzd\theta dr = \int_0^2\int_0^{\pi}\int_0^{r\sin\theta}\frac{z}{\sin\theta}dzd\theta dr =$$

   $$= \int_0^2\int_0^{\pi}\frac{r^2\sin\theta}{2}d\theta dr = \int_0^2 r^2 dr = \frac{8}{3}.$$

3. $\oint$16.6 2.

   (a) Th corresponding equation is

   $$r(\cos\theta+\sin\theta) = 1.$$

(b) The integral is equal to the volume of the cone with basis of radius 1 in the xy plane and altitude 1 intersected with the first octant. Hence in spherical coordinates, θ goes from 0 to $\frac{\pi}{2}$, φ goes from 0 to $\frac{\pi}{2}$, and ρ goes from 0 to $\frac{1}{\cos\varphi+\sin\varphi}$. Hence the integral is

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{\cos\varphi+\sin\varphi}} \rho^3 \sin\varphi\cos\varphi d\rho d\varphi d\theta.$$

4. $\oint$16.6 4.

Since $z \geq \sqrt{x^2+y^2}$, $\sin\varphi = \frac{\sqrt{x^2+y^2}}{\sqrt{z^2+x^2+y^2}} \leq \frac{\sqrt{2}}{2}$. Hence $0 \leq \varphi \leq \frac{\pi}{4}$. The moment of inertia is given by the integral:

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 \delta(r^2\sin^2\varphi)r^2\sin\varphi dr d\varphi d\theta =$$

$$= M\frac{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r^4\sin^3\varphi dr d\varphi d\theta}{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^1 r^2\sin\varphi dr d\varphi d\theta} =$$

$$= \frac{3M}{5}\frac{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin^3\varphi d\varphi d\theta}{\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\sin\varphi d\varphi d\theta} =$$

$$= \frac{3M}{5}\frac{2\pi\frac{8-5\sqrt{2}}{12}}{2\pi\frac{2-\sqrt{2}}{2}} = \frac{(3-\sqrt{2})M}{10}.$$

5. $\oint$17.2 1b,d.

(b)

$$\vec{R}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} = 3u\cos v\hat{i} + 2u\sin v\hat{j}.$$

Hence the partial derivatives of R are:

$$\left.\frac{\partial R}{\partial u}\right|_P = 3\cos v_0\hat{i} + 2\sin v_0\hat{j}, \quad \left.\frac{\partial R}{\partial v}\right|_P = -3u_0\sin v_0\hat{i} + 2u_0\cos v_0\hat{j},$$

and so the unit coordinate vectors are:

$$\hat{u}_P = \frac{3\cos v_0\hat{i} + 2\sin v_0\hat{j}}{\sqrt{5\cos^2 v_0 + 4}}, \quad \hat{v}_P = \frac{-3u_0\sin v_0\hat{i} + 2u_0\cos v_0\hat{j}}{\sqrt{u_0^2(4+5\sin^2 v_0)}}.$$

Since $\hat{u}_P \cdot \hat{v}_P \neq 0$, the system is not orthogonal.

(d) We easily get that $y = \frac{v}{u}$ and $x = \frac{u^2}{v}$.

$$\vec{R}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} = \frac{u^2}{v}\hat{\imath} + \frac{v}{u}\hat{\jmath}.$$

Hence the partial derivatives of R are:

$$\left.\frac{\partial R}{\partial u}\right|_P = \frac{2u_0}{v_0}\hat{\imath} - \frac{v_0}{u_0^2}\hat{\jmath}, \quad \left.\frac{\partial R}{\partial v}\right|_P = -\frac{u_0^2}{v_0^2}\hat{\imath} + \frac{1}{u_0}\hat{\jmath},$$

so the unit coordinate vectors are:

$$\hat{u}|_P = \frac{|v_0|}{v_0}\frac{2u_0^3\hat{\imath} - v_0^2\hat{\jmath}}{\sqrt{4u_0^6 + v_0^4}}, \quad \hat{v}|_P = \frac{|u_0|}{u_0}\frac{-u_0^3\hat{\imath} + v_0^2\hat{\jmath}}{\sqrt{u_0^6 + v_0^4}}.$$

Since, $\hat{u}|_P \cdot \hat{v}|_P \neq 0$, the system is not orthogonal.

6. $\oint$17.5 2.

We easily get that $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Hence

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix}\right\| = \frac{1}{2}.$$

Hence the integral is

$$\int\int R(x+y)^2 dA = \int_{-1}^{1}\int_{-1}^{1} \frac{u^2}{2} du dv = \int_{-1}^{1} \frac{1}{3} dv = \frac{2}{3}.$$

7. $\oint$17.5 4.

As we saw in problem 6,

$$u = x + y, \quad v = x - y, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

Hence

$$\int\int_R (x-y)^2(x+y)^4 dx dy =$$
$$= \int_{-1}^{1}\int_{-1}^{1} \frac{v^2u^4}{2} du dv = \int_{-1}^{1} \frac{v^2}{5} dv = \frac{2}{15}.$$

8. $\oint$17.5 5.

We take $u = x + y$ and $v = x - y$. As we saw in the problem 7,

$$u = x + y, \quad v = x - y, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

The curves $y = x$, $y = -x$, and $x + y = \cos(x - y)$ in the xy plane correspond to the curves $v = 0, u = 0$, and $u = \cos v$ in the uv plane. The area of the region will be

$$\int\int_R dxdy = \int_0^{\frac{\pi}{2}} \int_0^{\cos v} \frac{1}{2} dudv = \int_0^{\frac{\pi}{2}} \frac{\cos v}{2} dv = \frac{1}{2}.$$

9. ∮18.5 5,6.

   (5)
$$\vec{F}(x, y, z) = L(x, y, z)\hat{\imath} + M(x, y, z)\hat{\jmath} + N(x, y, z)\hat{k} =$$
$$= 2xyz\hat{\imath} + (x^2 z + 2yz)\hat{\jmath} + (x^2 y + y^2)\hat{k}.$$

   We have that

$$L_y = M_x = 2xz, \quad L_z = N_x = 2xy, \quad M_z = N_y = x^2 + 2y,$$

   so $\vec{F}$ passes the derivative test. If we take $f(x, y, z) = x^2yz + y^2z$, then $f_x = 2xyz = L, f_y = x^2z + 2yz = M, f_z = x^2y + y^2 = N$, so $\vec{F} = \vec{\nabla} f$, i.e. f is a scalar potential for $\vec{F}$.

   (6)
$$\vec{F}(x, y, z) = L(x, y, z)\hat{\imath} + M(x, y, z)\hat{\jmath} + N(x, y, z)\hat{k} =$$
$$= 4xy\hat{\imath} - x^2\hat{\jmath} + 4z\hat{k}.$$

   Then $L_y = 4x$ and $M_x = -2x$, so $L_y \neq M_x$, has $\vec{F}$ isn't a gradient field.

10. ∮18.6 1a,2b.

   (1a)
$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{z} =$$
$$= (\theta^2 + z^2)\hat{r} + 2\theta\hat{\theta} + 2rz\hat{z}.$$

   (2b)
$$\vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\hat{\varphi} + \frac{1}{\rho \sin\varphi}\frac{\partial f}{\partial \theta}\hat{\theta} =$$
$$= 2\rho\varphi \sin\theta\hat{\rho} + \rho \sin\theta\hat{\varphi} + \rho\varphi\frac{\cos\theta}{\sin\varphi}\hat{\theta}.$$

11. $\oint$18.7 2.

Since $r = \sqrt{x^2+y^2}$ and $\hat{r} = x\hat{\imath} + y\hat{\jmath}$, the expression for $\vec{F}$ in Cartesian coordinates is:

$$\vec{F} = (r + \frac{1}{r})\hat{r} = \frac{x^2+y^2+1}{x^2+y^2}(x\hat{\imath} + y\hat{\jmath}).$$

The vector $\vec{F}(P)$ will always point in the same direction as $\vec{O}P$, i.e. opposite from O. The magnitude of $\vec{F}(P)$ increases as OP goes to infinity and as OP goes to 0. The minimal magnitude is obtained on the circle of radius 1 and center O. If P and P' are on the same circle of center O, then $\vec{F}(P)$ and $\vec{F}(P')$ have equal magnitudes. Every ray starting at O is an integral curve. The general equation for integral curves is $\theta = a$ constant. If we take $f = \frac{x^2}{2} + \frac{y^2}{2} + \frac{\ln(x^2+y^2)}{2}$, then $f_x = x + \frac{x}{x^2+y^2} = \frac{x^2+y^2+1}{x^2+y^2}x$ and $f_y = \frac{x^2+y^2+1}{x^2+y^2}y$. Hence $\vec{F} = \vec{\nabla} f$, i.e. $\vec{F}$ is a gradient field, and f is a scalar potential for $\vec{F}$. The equipotential curves are the circles of center O. A general equation for equipotential curves is $x^2 + y^2 + \ln(x^2+y^2) = c$ constant, which is equivalent to the equation $x^2 + y^2 = c$ constant.