Lecture XII

Terminology for Point-Sets in Euclidean Spaces and Minimum-Maximum Theorems

First let us take a short look at a problem that was on the exam. We are given a level curve (in $\mathbf{E^2}$) or a surface(in $\mathbf{E^3}$) and a point P on that curve or surface. How to find a vector normal to that curve or surface at that point P? Let us consider the case of $\mathbf{E^3}$. The answer lies in the gradient of the function defining the surface. The graph of the surface is given by $z = f(x, y)$, hence $z - f(x, y) = g(x, y, z) = 0$. The gradient of g at P is a vector normal to the surface at P:

$$\vec{\nabla} g|_P = -\frac{\partial f}{\partial x}|_P \hat{\imath} - \frac{\partial f}{\partial y}|_P \hat{\jmath} + \hat{k}.$$

Now let us take a look at some basic notions of point-set topology of Euclidean spaces.

Definition 1 *Given a point P, we define a* neighborhood *of P in the following manner:*

(i) For the 1-dimensional space, a neighborhood is an interval of the form $[c - r, c + r]$ for some $r > 0$, where c is the point P.

(ii) For the 2-dimensional space, a neighorhood is a disc of points with center at P and radius r.

(iii) For the 3-dimensional space, a neighborhood is a solid ball of radius r and center P.

In all three cases, a neighborhood is a set U for which there exists $r \in \mathbb{R}$ such that U is the set of all points Q with $PQ \leq r$.

Let D be a given set of points.

Definition 2 *A point P is an* interior point *of D if there is some neighborhood of P which is entirely contained in D.*

Definition 3 *A point P is a* boundary point of D *if every neighborhood of P contains at least one point in D and at least one point not in D.*

Definition 4 *D is called* open *if every point in D is an interior point of D.*

D is called closed *if it contains all its boundary points.*

Note: D is open if and only if D contains none of its boundary points.

Definition 5 *D is* bounded *if there is some point P and some neighborhood of P such that the neighborhood contains the set D.*

Definition 6 *D is* compact *if D is closed and bounded.*

If f is a path for D, how do we find points P in D where f has a maximum value or a minimum value? We investigate this question, providing some methods of solving it. In doing so, we will consider the input(i.e. the points we are working with) to be of the form x, contained in the domain of f, and not of the form P, contained in D.

Definition 7 *Let f be defined on $[(a, b)]$ and let x be a point in its definition domain.*

(i) x is a global maximum point *if $f(x) \geq f(y)$ for all y in the definition domain.*

(ii) x is a global minimum point *if $f(x) \leq f(y)$ for all y in the definition domain.*

(iii) x is a local maximum point *if there exists a neighobrhood U of x such that $f(x) \geq f(y)$ for all y in intersection of U and the definition domain.*

(iv) x is a local minimum point *if there exists a neighobrhood U of x such that $f(x) \leq f(y)$ for all y in intersection of U and the definition domain.*

All maximum and minimum points, global and local, are called *extreme points of f*. The two following theorems are useful in finding extreme points.

Theorem 1 (Min-Max Existence Theorem) *Let f be a continuous function, having D as its domain. Assume D is compact.Then f has at least one global maximum point and at least one global minimum point in D.*

Definition 8 *Let f be a differentiable function. If $\vec{\nabla} f|_P = 0$, then P is called a* critical point.

Theorem 2 (Critical Point Theorem) *Let f be a differentiable function with a domain D and let P be an interior point of D. If P is an extreme point, then P is a critical point.*