Lecture XVIII

Change of Variables; Vector Fields

1 Change of Variables

Recall from lecture 17 that we change variables in integrals by the following formula:

$$\int\int_R f(x,y)dxdy = \int\int_{\hat{R}} f(x(u,v),y(u,v)) \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| dudv,$$

where R and $\hat{R}$ are corresponding regions in the xy and uv planes.

Changing variables lets us easily compute integrals. For example, let us find the area in $\mathbf{E^2}$ enclosed by the curves $xy = 1, xy = 3$, $xy^2 = 1$, and $xy^2 = 2$ in the first quadrant. Denote the region enclosed by these curves by R. If we choose new coordinates $u = xy$ and $v = xy^2$, the region $\hat{R}$ corresponding to R in the uv plane is the rectangle of vertices $(1,1)$, $(3,1)$, $(3,2)$, and $(1,2)$. Changing variables to u, v, the area of R is given by

$$\int\int_R dxdy = \int_1^2 \int_1^3 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv = \int_1^2 \int_1^3 1 \Bigg/ \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dudv =$$

$$= \int_1^2 \int_1^3 1 \Bigg/ \begin{vmatrix} y & x \\ y^2 & 2xy \end{vmatrix} dudv = \int_1^2 \int_1^3 \frac{1}{xy^2} dudv =$$

$$= \int_1^2 \int_1^3 \frac{1}{v} dudv = \int_1^2 \frac{2}{v} dv = 2\ln 2.$$

2 Vector Fields

Definition 1 *A* vector field *in* $\mathbf{E^2}$ *is a function defined on a region* R *in* $\mathbf{E^2}$ *that gives as outputs vectors in* $\mathbf{E^2}$.

Clearly all vector operations apply to vector fields as well. We can write a vector field as following:

$$\vec{F}(x,y) = f(x,y)\hat{\imath} + g(x,y)\hat{\jmath}.$$

An important category of vector fields that we will be working with is the category of *gradient fields.* We will see that not all vector fields are gradient fields.

Definition 2 *Let $\vec{F}(x,y)$ be a vector field in $\mathbf{E^2}$. If there exists a scalar field f such that*

$$\vec{\nabla} f_P = \vec{F}(P) \quad for \quad all \quad P,$$

then $\vec{F}$ is called a gradient field. *Also, f is called a* scalar potential for $\vec{F}$*.*

The following lemma, which helps us determine if a vector field is a gradient field, is called the *derivative test for a gradient field.*

Lemma 1 *Let $\vec{F}$ be a vector field, $\vec{F}(x,y) = f(x,y)\hat{\imath} + g(x,y)\hat{\jmath}$. F can be a gradient field only if $f_y = g_x$.*

This test doesn't always tell us if F is a gradient filed or not, it only states that if the condition $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ isn't satisfied, then we can be sure that F isn't a gradient field.

Proof of Lemma 1:

Let $\vec{F} = f\hat{\imath} + g\hat{\jmath}$ be a gradient field. Then there exists f such that

$$\vec{F} = \vec{\nabla} f = h_x\hat{\imath} + h_y\hat{\jmath}.$$

Hence $f = h_x$ and $g = h_y$, and since $(h_x)_y = (h_y)_x$, we get that $f_y = g_x$.