Lecture XIII

Two-Variable Test

Constrained Maximum-Minimum Problems

1 The two-variable test

Recall that by the Critical Point Theorem, onlyif the gradient of a function f at P is 0 (i.e. P is a critical point for f), can P be an extreme point of f.

Let $f(x,y) = x^3 + xy + y^3$. Then

$$\vec{\nabla} f = (3x^2 + y)\hat{\imath} + (x + 3y^2)\hat{\jmath},$$

so if $P = (x,y)$ is a critical point, then

$$x = -3y^2, \quad y = -3x^2,$$

hence $x + 27x^4 = 0$, so the only critical points are $P_1(0,0)$ and $P_2(-\frac{1}{3}, -\frac{1}{3})$. Now we have to find out if these points are extreme points. For this we can use the following test.

Theorem 1 (The two-variable test) *Let f be a two-variable C^2 function, and let P be an interior critical point for f. We define the functions H_1 and H_2, called* Hessian functions *in the following manner:*

$$H_1(P) = f_{xx}(P),$$

$$H_2(P) = \begin{vmatrix} f_{xx}(P) & f_{xy}(P) \\ f_{yx}(P) & f_{yy}(P) \end{vmatrix} = f_{xx}(P)f_{yy}(P) - f_{xy}^2(P)$$

Then:

(a) If $H_1(P) > 0$ and $H_2(P) > 0$ then P is a local minimum point for f.

(b) If $H_1(P) < 0$ and $H_2(P) > 0$ then P is a local maximum point for f.

(c) If $H_2(P) < 0$ then P is a saddle point for f.

Going back to the function $f(x, y) = x^3 + xy + y^3$, we find that $H_1(x, y) = 6x$ and $H_2(x, y) = 36xy - 1$. Hence $H_1(P_1) = 0, H_2(P_1) = -1$, so $P_1(0, 0)$ is a saddle point. Also, $H_1(P_2) = -2$ and $H_2(P_2) = 3$, so $P_2(-\frac{1}{3}, -\frac{1}{3})$ is a local maximum point.

2 Constrained problems

Suppose f is a function defined on a region D in $\mathbf{E^3}$ and C is a surface or a curve in D. We can think of C as a new domain for f and try to find extreme point for f in C. Such a problem is a called a *constrained problem*.

Let f and g be C^1 and let D' be the set of all points P such that $g(P) = c$. Assume $\vec{\nabla} f|_P \neq 0$ at all points P in D'. Let D' be the new restricted domain of f. Such a case is called a one constraint case, since the restricted domain is given by one function, g.

Definition 1 *Let P be a point in D'. We call P a* constrained critical point for f on D' *if there is a scalar λ^P such that $\vec{\nabla} f|_P = \lambda^P \vec{\nabla} g|_P$.*

Note that for a constrained critical point P, if $\vec{\nabla} f|_P \neq 0$, then $\vec{\nabla}|_P$ is normal to D' at P.

Theorem 2 *If P is a extreme point for f on D', then P must be a constrained critical point for f on D'.*

The one-constraint Lagrange method of finding extreme points on D' uses the theorem above in the following manner. First we define a function h on D', $h(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$. Then we form the system of equations

$$h_x(x, y, z, \lambda) = 0, \quad h_y(x, y, z, \lambda) = 0, \quad h_z(x, y, z, \lambda) = 0, \quad g(x, y, z) = c$$

and solve it. Its solutions are the constrained critical points of f. Finally, among these points we identify the extreme points for f on D'.

The case of two constraints can be described as follows. Let f, g_1 and g_2 be C^1 functions. Let D' be the set of all points P such that $g_1(P) = c_1$ and $g_2(P) = c_2$. Assume $\vec{\nabla} g_1|_P \times \vec{\nabla} g_2|_P \neq 0$ for all points P in D'.

Definition 2 *Let P be a point in D'. We call P a* constrained critical point for f on D' *if there exist scalars λ_1^P and λ_2^P such that $\vec{\nabla}|_P = \lambda_1^P \vec{\nabla} g_1|_P + \lambda_2^P \vec{\nabla} g_2|_P$.*

Theorem 3 *If P is a extreme point for f on D', then P must be a constrained critical point for f on D'.*

The Lagrange method for two constraints is similar to the method for one constraint. We define $h(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) - \lambda_1 g_1(x, y, z) - \lambda_2 g_2(x, y, z)$. We form the system of equations

$$h_x(x, y, z, \lambda_1, \lambda_2) = 0, \quad h_y(x, y, z, \lambda_1, \lambda_2) = 0, \quad h_z(x, y, z, \lambda_1, \lambda_2) = 0,$$

$$g_1(x, y, z) = c_1 \quad g_2(x, y, z) = c_2$$

and solve it. The solutions will be the constrained critical points. From among these points, we identify the extreme points of f on D'.