

## Problem Set III Solutions

1. §6.2 2.

Denote  $\theta = T\hat{O}Q$ . The unwound portion of the string has the same length as the arc  $TQ$  in clockwise direction from  $T$  to  $Q$ , so  $TP = a\theta$ . Also,  $T$  has coordinates  $(a \cos \theta, a \sin \theta)$ . Since  $TP$  is perpendicular to  $TO$  and the string is unwound counter-clockwise,  $T\vec{P} = a\theta(\cos(\theta - \frac{\pi}{2})\hat{i} + \sin(\theta - \frac{\pi}{2})\hat{j}) = a\theta(\sin \theta \hat{i} - \cos \theta \hat{j})$ . So the parametric equations for  $T$  are  $x = a \cos \theta + a\theta \sin \theta$  and  $y = a \sin \theta - a\theta \cos \theta$ .

2. §6.3 2.

$\frac{d\vec{R}(t)}{dt} = (\cos t - t \sin t)\hat{i} + (\sin t + t \cos t)\hat{j} + \hat{k}$ . Then

$$\left| \frac{d\vec{R}(t)}{dt} \right| = \sqrt{\cos^2 t + \sin^2 t + t^2 \sin^2 t + t^2 \cos^2 t + 1} = \sqrt{t^2 + 2}.$$

Hence  $\hat{T}(t) = \left( \frac{\cos t - t \sin t}{\sqrt{t^2 + 2}} \right) \hat{i} + \left( \frac{\sin t + t \cos t}{\sqrt{t^2 + 2}} \right) \hat{j} + \left( \frac{1}{\sqrt{t^2 + 2}} \right) \hat{k}$ .

3. §6.4 3.

Let  $\vec{R}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ . For any function  $f(t)$  we will denote by  $f'(t)$  the first derivative of  $f$  with respect to  $t$  and by  $f''(t)$  the second derivative with respect to  $t$ . Since  $\frac{ds}{dt} = \left| \frac{d\vec{R}(t)}{dt} \right| = 1$ , it follows that  $(f_1')^2(t) + (f_2')^2(t) = 1$ . But by the definition of the curve,  $f_1(t) = x$  and  $f_2(t) = y = e^x$ , hence  $f_2'(t) = f_1'(t)e^{f_1(t)}$ . Then  $(f_1')^2(t) = (1 + e^{2f_1(t)})^{-1}$ , so  $f_1'(t) = (1 + e^{2f_1(t)})^{-\frac{1}{2}} = (1 + e^{2x})^{-\frac{1}{2}}$ . Also,  $f_2'(t) = f_1'(t)e^{f_1(t)} = (1 + e^{2x})^{-\frac{1}{2}} e^x$ .

So  $\vec{v} = f_1'(t)\hat{i} + f_2'(t)\hat{j} = (1 + e^{2x})^{-\frac{1}{2}}\hat{i} + (1 + e^{2x})^{-\frac{1}{2}} e^x \hat{j}$ .

From  $(f_1')^2(t) = (1 + e^{2f_1(t)})^{-1}$ , it follows that  $2f_1''(t)f_1'(t) = (1 + e^{2f_1(t)})'$ .  $(-1)(1 + e^{2f_1(t)})^{-2} = -2f_1''(t)e^{2f_1(t)}(1 + e^{2f_1(t)})^{-2}$ , hence  $f_1''(t) = -e^{2x}(1 + e^{2x})^{-2}$ . Also,  $f_2''(t) = (e^{f_1(t)})'' = e^{f_1(t)}(f_1''(t) + (f_1')^2(t)) = e^x(1 + e^{2x})^{-2}$ .

So  $\vec{a} = f_1''(t)\hat{i} + f_2''(t)\hat{j} = (-e^{2x}(1 + e^{2x})^{-2})\hat{i} + e^x(1 + e^{2x})^{-2}\hat{j}$ .

4. §6.4 4.

- (a) It is quite clear that we can choose  $x = 2 \cos \theta$ ,  $y = 3 \sin \theta$ , for  $\theta$  from 0 to  $2\pi$ .
- (b)  $\vec{R}(\theta) = 2 \cos \theta \hat{i} + 3 \sin \theta \hat{j}$ , so  $\frac{d\vec{R}(\theta)}{d\theta} = -2 \sin \theta \hat{i} + 3 \cos \theta \hat{j}$ . Hence the length  $s$  of the curve is given by the integral

$$s = \int_0^{2\pi} \sqrt{4 + 9 \cos^2 \theta} d\theta.$$

5. §6.7 8.

- (a)  $\dot{s} = |\vec{v}| = \sqrt{3}$
- (b)  $\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$
- (c)  $k = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{|\hat{i} - \hat{j}|}{3\sqrt{3}} = \frac{\sqrt{2}}{3\sqrt{3}}$
- (d) Using (e):  $\vec{a} = \ddot{s}\hat{T} + k\dot{s}^2\hat{N}$ , so  $\hat{k} = \frac{1}{\sqrt{3}}\ddot{s}(\hat{i} + \hat{j} + \hat{k}) + \frac{1}{3}(-\hat{i} - \hat{j} + 2\hat{k})$ .  
Hence  $\ddot{s} = \frac{\sqrt{3}}{3}$ .
- (e)  $\hat{N} = \frac{(\vec{v} \times \vec{a}) \times \vec{v}}{|\vec{v} \times \vec{a}| |\vec{v}|} = \frac{(\hat{i} - \hat{j}) \times (\hat{i} + \hat{j} + \hat{k})}{\sqrt{2}\sqrt{3}} = \frac{1}{\sqrt{6}}(-\hat{i} - \hat{j} + 2\hat{k})$

6. §7.4 2.

- (a)  $x = r(t) \cos \theta(t) = t$ ,  $y = r(t) \sin \theta(t) = t$ ,  $z = z(t) = t$
- (b)  $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} = \sqrt{2}\hat{r} + \hat{z}$

7. §7.5 4.

- (a) We have that  $\dot{r}(t) = 1$ ,  $\dot{\theta}(t) = 2t$ , and  $\dot{z}(t) = 3t^2$ .  
Hence  $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} = \hat{r} + 2t^2\hat{\theta} + 3t^2\hat{z}$ .
- (b) We have that  $\ddot{r}(t) = 0$ ,  $\ddot{\theta}(t) = 2$ , and  $\ddot{z}(t) = 6t$ .  
Hence  $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} + \ddot{z}\hat{z} = -4t^3\hat{r} + 6t\hat{\theta} + 6t\hat{z}$ .
- (c) Going to Cartesian coordinates,  $\vec{R}(t) = t \cos t^2 \hat{i} + t \sin t^2 \hat{j} + t^3 \hat{k}$ . Then  $\frac{d\vec{R}}{dt} = (\cos t^2 - 2t^2 \sin t^2)\hat{i} + (\sin t^2 + 2t^2 \cos t^2)\hat{j} + 3t^2\hat{k}$ . Hence  $\frac{ds}{dt} = \left| \frac{d\vec{R}}{dt} \right| = \sqrt{1 + 13t^4}$ .
- (d) The arc length is given by the integral  $\int_0^5 \sqrt{1 + 13t^4} dt$ .

8. §7.5 8.

(a) Since  $\dot{r} = 2t$ ,  $\dot{\theta} = 1$ , and  $\dot{z} = 1$ , it follows that  $\vec{v}(t) = 2t\hat{r} + t^2\hat{\theta} + \hat{z}$ , so  $\vec{v}(1) = 2\hat{r} + \hat{\theta} + \hat{z}$ .

Since  $\ddot{r} = 2$ ,  $\ddot{\theta} = 0$ , and  $\ddot{z} = 0$ , it follows that  $\vec{a}(t) = (2 - t^2)\hat{r} + 4t\hat{\theta}$ , so  $\vec{a}(1) = \hat{r} + 4\hat{\theta}$ .

(b) Since  $\vec{v}(1) \times \vec{a}(1) = -4\hat{r} + \hat{\theta} + 7\hat{z}$ ,  $|\vec{v}(1) \times \vec{a}(1)| = \sqrt{66}$ . Also,  $|\vec{v}(1)| = \sqrt{6}$ . Hence  $k(1) = \frac{|\vec{v}(1) \times \vec{a}(1)|}{|\vec{v}(1)|^3} = \frac{\sqrt{66}}{6\sqrt{6}} = \frac{\sqrt{11}}{6}$ .

9. §7.5 12.

Since  $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$ , we have that  $\ddot{\hat{\theta}} = -\ddot{\theta}\hat{r} - \dot{\theta}\dot{\hat{r}}$ . Since  $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ ,  $\ddot{\hat{\theta}} = -(\ddot{\theta}\hat{r} + \dot{\theta}^2\hat{\theta})$ .

Then  $\ddot{\hat{\theta}} = -(\ddot{\theta}\hat{r} + \dot{\theta}\dot{\hat{r}} + 2\dot{\theta}\ddot{\theta}\hat{\theta} + \dot{\theta}^2\dot{\hat{\theta}})$ . Since  $\ddot{\theta}\hat{r} = \ddot{\theta}\hat{\theta}$  and  $\dot{\hat{\theta}} = -\dot{\theta}\hat{r}$ , it follows that  $\ddot{\hat{\theta}} = (\dot{\theta}^3 - \ddot{\theta})\hat{r} - 3\dot{\theta}\ddot{\theta}\hat{\theta}$ .

10. §8.1 2.

We consider  $O$  to be the origin of the Cartesian system, and  $M$  the plane  $xy$ . If  $P$  has coordinates  $(x, y, z)$ , then its projection on  $M$  has coordinates  $(x, y, 0)$ , so the distance from  $O$  to the projection of  $P$  on  $M$  is  $\sqrt{x^2 + y^2}$ . Hence  $f(P) = \sqrt{x^2 + y^2}$ .

11. §8.6 2ad.

(a)

$$\begin{aligned} \frac{\partial f}{\partial x} &= ye^x, & \frac{\partial f}{\partial y} &= e^x \\ \frac{\partial^2 f}{\partial x^2} &= ye^x, & \frac{\partial^2 f}{\partial y \partial x} &= e^x, & \frac{\partial^2 f}{\partial x \partial y} &= e^x, & \frac{\partial^2 f}{\partial y^2} &= 0 \end{aligned}$$

(d)

$$\begin{aligned} \frac{\partial f}{\partial x} &= -2xe^{-x^2-y}, & \frac{\partial f}{\partial y} &= -e^{-x^2-y} \\ \frac{\partial^2 f}{\partial x^2} &= (4x^2 - 2)e^{-x^2-y}, & \frac{\partial^2 f}{\partial y \partial x} &= 2xe^{-x^2-y} \\ \frac{\partial^2 f}{\partial x \partial y} &= 2xe^{-x^2-y}, & \frac{\partial^2 f}{\partial y^2} &= e^{-x^2-y} \end{aligned}$$

12. §8.6 7.

From the determinant we get  $q = -13x - yz + 3z$ . Then  $\frac{\partial q}{\partial x} = -13$ ,  $\frac{\partial q}{\partial y} = -z$ ,  $\frac{\partial q}{\partial z} = -y + 3$ .

- (a) If  $P(x, y, z) = (0, 0, 0)$ , then  $\frac{\partial q}{\partial x}|_P = -13$ ,  $\frac{\partial q}{\partial y}|_P = 0$ ,  $\frac{\partial q}{\partial z}|_P = 3$ , so  $q$  is going to be most sensitive to changes in the  $x$  variable.
- (b) If  $P(x, y, z) = (10, 15, 20)$ , then  $\frac{\partial q}{\partial x}|_P = -13$ ,  $\frac{\partial q}{\partial y}|_P = -20$ ,  $\frac{\partial q}{\partial z}|_P = -12$ , so  $q$  is going to be most sensitive to changes in the  $y$  variable.
- (c) If  $P(x, y, z) = (15, 15, 15)$ , then  $\frac{\partial q}{\partial x}|_P = -13$ ,  $\frac{\partial q}{\partial y}|_P = -15$ ,  $\frac{\partial q}{\partial z}|_P = -12$ , so  $q$  is going to be most sensitive to changes in the  $y$  variable.

13. §8.6 8.

- (a) We have that  $V = \pi a^2 b$ . Then  $\frac{\partial V}{\partial a} = 2\pi ab$  and  $\frac{\partial V}{\partial b} = \pi a^2$ . If  $a = 0$ ,  $V$  is equally sensitive to changes in  $a$  and  $b$ . If  $a \neq 0$ ,  $V$  is more sensitive to changes in  $a$  if  $|b| > \frac{|a|}{2}$ .  $V$  is more sensitive to changes in  $b$  if  $|b| < \frac{|a|}{2}$  and  $V$  is equally sensitive to changes in  $a$  and  $b$  if  $|b| = \frac{|a|}{2}$ .
- (b) We have that  $S = 2\pi ab + 2\pi a^2$ . Then  $\frac{\partial S}{\partial a} = 2\pi(b + 2a)$  and  $\frac{\partial S}{\partial b} = 2\pi a$ . Hence,  $S$  is more sensitive to changes in  $a$  if  $|b + 2a| > |a|$ ,  $S$  is more sensitive to changes in  $b$  if  $|b + 2a| < |a|$ , and  $S$  is equally sensitive to changes in  $a$  and  $b$  if  $|b + 2a| = |a|$ .