

Problem Set X Solutions

1. § 24.3 2.

On C_1 , we take $\vec{R}(t) = 3 \cos t \hat{i} + 3 \sin t \hat{j}$ with $0 \leq t \leq \pi$, so $\frac{d\vec{R}}{dt} = -3 \sin t \hat{i} + 3 \cos t \hat{j}$. On C_2 we take $\vec{R}(t) = 3t \hat{i}$, with $-1 \leq t \leq 1$. Then

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{R} &= \int_{C_2} \vec{F} \cdot d\vec{R} + \int_{C_1} \vec{F} \cdot d\vec{R} = \\ &= \int_{-1}^1 3e^{9t^2} dt + \int_0^\pi -3 \sin t e^{9 \cos^2 t} dt + \\ &\quad + \int_0^\pi (-27 \sin^2 t + 45 \cos^2 t - 3 \cos t e^{-9 \sin^2 t}) dt = \\ &= \int_0^\pi -27 \sin^2 t + 45 \cos^2 t = -27\pi + 72 \int_0^\pi \cos^2 t = 9\pi, \end{aligned}$$

since $\int_{-1}^1 e^{9t^2} dt = \int_0^\pi \sin t e^{9 \cos^2 t} dt$ and $\int_0^\pi \cos t e^{-9 \sin^2 t} dt = 0$.

2. § 24.3 3.

The ellipse can be parametrized by $\vec{R}(r, \theta) = ra \cos \theta \hat{i} + rb \sin \theta \hat{j}$, with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Then $\frac{d\vec{R}}{dr} = a \cos \theta \hat{i} + b \sin \theta \hat{j}$ and $\frac{d\vec{R}}{d\theta} = -ra \sin \theta \hat{i} + rb \cos \theta \hat{j}$. Hence $\vec{w}(r, \theta) = rab \hat{k}$, so the area of the ellipse is

$$\iint_R dA = \int_0^{2\pi} \int_0^1 |\vec{w}| dr d\theta = 2\pi ab \int_0^1 r dr = \pi ab$$

3. § 24.3 4.

By the Green theorem, the area is given by the integral $\frac{1}{2} \int_C (-y \hat{i} + x \hat{j}) \cdot d\vec{R}$, where C is the reunion of the cycloid and the segment on the x axis. Let C_1 be the segment on the x axis. Its parametrization is $\vec{R}(t) = a(t - \sin t) \hat{i}$, with $2k\pi \leq t \leq (2k+2)\pi$. Hence on C_1 $\vec{F} = -y \hat{i} + x \hat{j} = a(t - \sin t) \hat{j}$, so $\int_{C_1} \vec{F} \cdot d\vec{R} = 0$. Let C_2 be the cycloid segment. Its parametrization is $\vec{R}(t) = a(t - \sin t) \hat{i} + a(1 - \cos t) \hat{j}$. Hence $\frac{d\vec{R}}{dt} = a(1 - \cos t) \hat{i} + a \sin t \hat{j}$ and $\vec{F} = -a(1 - \cos t) \hat{i} + a(t - \sin t) \hat{j}$. Hence

$$\int_{C_2} \vec{F} \cdot d\vec{R} = \int_{(2k+2)\pi}^{2k\pi} a^2 (t \sin t - \sin^2 t - 1 + 2 \cos t - \cos^2 t) dt =$$

$$\begin{aligned}
&= a^2 \int_{2k\pi}^{(2k+2)\pi} (2 - 2 \cos t - t \sin t) dt = \\
&= 4\pi a^2 - a^2 \int_{2k\pi}^{(2k+2)\pi} t \sin t dt = 2\pi a^2
\end{aligned}$$

So the area is

$$\begin{aligned}
&\frac{1}{2} \oint_C (-y\hat{i} + x\hat{j}) \cdot d\vec{R} = \\
&= \frac{1}{2} \int_{C_1} \vec{F} \cdot d\vec{R} + \frac{1}{2} \int_{C_2} \vec{F} \cdot d\vec{R} = \frac{1}{2} 2\pi a^2 = \pi a^2
\end{aligned}$$

4. § 24.3 7.

Since C is clockwise, we have that $\oint_C \vec{F} \cdot d\vec{R} = - \int \int_R \text{rot} \vec{F} dA$. Since $\text{rot} \vec{F} = M_x - L_y = y - 1$, the integral is

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{R} &= - \int \int_R \text{rot} \vec{F} dA = - \int_0^1 \int_0^1 (y - 1) dy dx = \\
&= \int_0^1 (1 - y) dy = \frac{1}{2}
\end{aligned}$$

5. § 24.5 1.

(a) Denote C_1 the counterclockwise circle of radius 3 with center at O .

Then from 24.5.1 it follows that

$$\oint_C \vec{F} \cdot d\vec{R} = - \oint_{C_1} \vec{F} \cdot d\vec{R} = -4$$

(b) Since C does not contain O , we can apply Green's theorem and get

$$\oint_C \vec{F} \cdot d\vec{R} = 0$$

(c) From 24.5.1 it follows that

$$\oint_C \vec{F} \cdot d\vec{R} = - \oint_{C_1} \vec{F} \cdot d\vec{R} = -4$$

(d) If C does not contain O , $\int_C \vec{F} \cdot d\vec{R} = 0$. If C contains O , then if it is clockwise, we can take a counterclockwise curve C_2 containing both C and C_1 and get that

$$\int_C \vec{F} \cdot d\vec{R} = - \int_{C_2} \vec{F} \cdot d\vec{R} = - \int_{C_1} \vec{F} \cdot d\vec{R} = -4$$

In the same manner, if the curve contains O and it is counterclockwise, $\int_C \vec{F} \cdot d\vec{R} = 4$. So the possible values are $0, \pm 4$.

(e) Using the same reasoning as in (d), depending on the winding number of C , the possible values of the integral are of the form $4n$, with n integer.

6. § 24.5 2.

(a) If the circle is counterclockwise, $\oint_C \vec{F} \cdot d\vec{R} = -5$. If the circle is clockwise, $\oint_C \vec{F} \cdot d\vec{R} = 5$.

(b) The possible values of $\oint_C \vec{F} \cdot d\vec{R}$ are $0, \pm 5$.

7. § 24.7 2.

Since $x = r \cos \theta$ and $y = r \sin \theta$, we have that

$$dx = \cos \theta dr - r \sin \theta d\theta,$$

$$dy = \sin \theta dr + r \cos \theta d\theta$$

Hence

$$dr = (\cos^2 \theta + \sin^2 \theta) dr = \cos \theta dx + \sin \theta dy,$$

so $\frac{\partial r}{\partial x} = \cos \theta$ and $\frac{\partial r}{\partial y} = \sin \theta$. Also

$$r d\theta = r(\cos^2 \theta + \sin^2 \theta) d\theta = -\sin \theta dx + \cos \theta dy,$$

so $\frac{\partial \theta}{\partial x} = \frac{-\sin \theta}{r}$ and $\frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r}$.

8. § 24.7 3.

(1) $\vec{F} = r\hat{r}$, so $G(r, \theta) = r$, $H(r, \theta) = 0$. Hence $\frac{\partial G}{\partial \theta} = 0$ and $\frac{\partial rH}{\partial r} = 0$. So

$$\text{rot} \vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = 0$$

(2) $\vec{F} = f(r)\hat{r}$, so $G(r, \theta) = f(r)$, $H(r, \theta) = 0$. Hence $\frac{\partial G}{\partial \theta} = 0$ and $\frac{\partial rH}{\partial r} = 0$. So

$$\text{rot} \vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = 0$$

(3) $\vec{F} = r\hat{\theta}$, so $G(r, \theta) = 0$, $H(r, \theta) = r$. Hence $\frac{\partial G}{\partial \theta} = 0$ and $\frac{\partial rH}{\partial r} = 2r$.

So

$$\text{rot} \vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = 2$$

(4) $\vec{F} = r\hat{\theta}$, so $G(r, \theta) = 0$, $H(r, \theta) = \frac{1}{r}$. Hence $\frac{\partial G}{\partial \theta} = 0$ and $\frac{\partial rH}{\partial r} = 0$. So

$$\text{rot}\vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = 0$$

(5) $\vec{F} = r\hat{\theta}$, so $G(r, \theta) = r\theta$, $H(r, \theta) = r$. Hence $\frac{\partial G}{\partial \theta} = r$ and $\frac{\partial rH}{\partial r} = 2r$.

So

$$\text{rot}\vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = 1$$

(6) $\vec{F} = r\hat{\theta}$, so $G(r, \theta) = 3r^2\theta$, $H(r, \theta) = r^2$. Hence $\frac{\partial G}{\partial \theta} = 3r^2$ and $\frac{\partial rH}{\partial r} = 3r^2$. So

$$\text{rot}\vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = 0$$

(7) $\vec{F} = r\hat{\theta}$, so $G(r, \theta) = 3r^2$, $H(r, \theta) = -r^2$. Hence $\frac{\partial G}{\partial \theta} = 3r^2$ and $\frac{\partial rH}{\partial r} = -3r^2$. So

$$\text{rot}\vec{F} = \frac{1}{r} \frac{\partial}{\partial r}(rH) - \frac{1}{r} \frac{\partial G}{\partial \theta} = -6r$$