Lecture XXIII

Surface Integrals

Remember that the parametric expression for a curve is given by a vector function $\vec{R}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. The parametric expression for a surface S is given by a vector function of two variables: $\vec{R}(u,v) = x(u,v)\hat{\imath} + y(u,v)\hat{\jmath} + z(u,v)\hat{k}$. The following are two fundamental properties of the parametric normal vector $\vec{w}(u,v)$:

1. If $\vec{w}_P \neq 0$, then $\vec{w}_P$ is normal to S at P.
2. $|\vec{w}_P|$ is an amplification factor from the area $\hat{U}$ around $\hat{P}$ to the surface area U around P.

We can use the Jacobian to find the paramteric nromal vector:

$$\vec{w}(u,v) = \frac{d\vec{R}}{du}\Big|_P \times \frac{d\vec{R}}{dv}\Big|_P = \frac{\partial(y,z)}{\partial(u,v)}\hat{\imath} + \frac{\partial(x,z)}{\partial(u,v)}\hat{\jmath} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}.$$

Consider the semicylinder S of radius a, with $0 \leq z \leq b$ and $y \geq 0$. In cylindrical coordinates, it is described by $0 \leq \theta \leq \pi$, $0 \leq z \leq b$, $r = a$. Clearly, since $\vec{w}_P$ is normal to S, $\frac{\vec{w}_P}{|\vec{w}_P|} = \hat{r}$. Since Δz and $\Delta\theta$ in the $z\theta$ plane correspond to Δz and $a\Delta\theta$ on the semicylinder, $|\vec{w}_P| = a$. Hence $\vec{w}_P = a\hat{r}$. We now give several definitions concerning types of surfaces.

Definition 1 *A surface is said to be* elementary *if its representation $\vec{R}$ is a one-to-one mapping.*

Definition 2 *A surface is* finite *if it can be divided into a finite number of elementary surfaces, where any two of these surfaces intersect at most along their common boundary, and no segment of boundary is contained in more than two of these surfaces.*

Definition 3 *A finite surface S is* one-sided *if there exists a continuous loop on S such that if we start from a point P and consider the parametric normal* *vector on the loop, when arriving at P again the vector has direction opposite of the one from the start. If such a loop does not exist, then we say the surface is* two-sided.

Definition 4 *A finite surface that has no boundary is called* closed.

Definition 5 *Let D be a connected region in $\mathbf{E^3}$. D is* simply-connected *if for every loop C contained in D, there exists a two-sided surface S such that C is the boundary of S and S is contained in D.*

Definition 6 *Let D be a connected region in $\mathbf{E^3}$. D is* two-connected *if for every closed surface S in D, the interior of S is also contained in D.*

We will now define surface integrals. There are scalar and vector surface integrals.

Definition 7 *Let f be a scalar field defined on a finite surface S. The scalar surface integral of f on S is denoted by $\int\int_S f d\sigma$ and is the limit of a Riemann sum:*

$$\int\int_S f d\sigma = \lim_{\substack{n\to\infty \\ \max d_i \to 0}} \sum_{i=1}^{n} f(P_i^*)\Delta\sigma_i,$$

where $\Delta S_1, \ldots, \Delta S_n$ form a subdivision of S intro elementary surfaces, P_i^* is a point in ΔS_i, $\Delta\sigma_i$ is the surface area of ΔS_i, and d_i is the diameter of ΔS_i for all $i \le n$.

Definition 8 *A* directed surface *S is a two-sided surface with all its nonzero normal vectors pointing away from the same side. The normal vectors are said to give a* direction *to S.*

Definition 9 *Let $\vec{F}$ be a vector field defined on a finite directed surface S. The vector surface integral of $\vec{F}$ on S is denoted by $\int\int_S (\vec{F}\cdot\hat{n})d\sigma$ and is the limit of a Riemann sum:*

$$\int\int_S (\vec{F}\cdot\hat{n})d\sigma = \lim_{\substack{n\to\infty \\ \max d_i \to 0}} \sum_{i=1}^{n} [\vec{F}(P_i^*)\cdot\hat{n}(P_i^*)]\Delta\sigma_i,$$

where $\Delta S_1, \ldots, \Delta S_n$ form a subdivision of S intro elementary surfaces, P_i^* is a point in ΔS_i, $\Delta\sigma_i$ is the surface area of ΔS_i, d_i is the diameter of ΔS_i, and $\hat{n}(P_i^*)$ is the normal vector on S at P_i^* for all $i \le n$.

We also denote the vector surface integral by $\int\int_S \vec{F} \cdot d\vec{\sigma}$. To evaluate surface integrals, the following formula are useful:

$$\int\int_S f d\sigma = \int\int_{\hat{D}} f(x(u,v), y(u,v), z(u,v))|\vec{w}(u,v)|dudv$$

$$\int\int_S \vec{F} \cdot d\vec{\sigma} = \int\int_{\hat{D}} \vec{F} \cdot \vec{w}(u,v)dudv$$