Lecture XI

Chain Rule: Elimination Method

Let $w = f(x, y)$ be a differentiable function of x and y. The linear approximation of f is given by

$$\Delta f_{app} = f_x(x, y)\Delta x + f_y(x, y)\Delta y.$$

We introduce a new notation, the *differential notation* for the increments Δf, Δx, Δy, namely we write df, dx, dy instead: $df = f_x dx + f_y dy$. This expression is called the *differential of f*. For example, the differential of

$$w = x^2 + y^2 - 1 \quad is \quad dw = 2xdx + 2ydy.$$

For any function $w = f(x, y)$, the equality

$$dw = \left(\frac{\partial w}{\partial x}\right)_y dx + \left(\frac{\partial w}{\partial y}\right)_x dy$$

holds. By the *elimination method*, we can find $(\frac{\partial w}{\partial x})_y$ and $(\frac{\partial w}{\partial y})_x$ if w is not given directly as a function of x and y but can be reduced to such a function. We ilustrate this in the following example. Consider the these two equalities:

$$w = f(x, y, z) = xyz, \qquad z = g(x, y) = e^{xy}.$$

Then the differentials of w and z are

$$dw = yzdx + xzdy + xydz \qquad and \qquad dz = ye^{xy}dx + xe^{xy}dy.$$

Substituting dz in the first equality, we get

$$dw = (yz + xy^2e^{xy})dx + (xz + yx^2e^{xy})dy.$$

Then the derivative of w with respect to x when w is seen as a function of x and y is precisely th term of dx in the equality above:

$$\left(\frac{\partial w}{\partial x}\right)_y = yz + xy^2e^{xy} \qquad and \qquad \left(\frac{\partial w}{\partial y}\right)_x = xz + yx^2e^{xy}.$$