Lecture XX

Vector Line Integrals; Conservative Fields

1 Vector line integrals

Let $\vec{F}$ be a vector field of domain D. Let D be connected, i.e. for any two points P, P' in D there is a curve C contained in D that goes from P to P'. Let C be a finite directed curve contained in D that is rectifiable, i.e. it has a length. We define the vector line integral $\int_C \vec{F} \cdot d\vec{R}$ as the limit of a Riemann sum:

$$\int_C \vec{F} \cdot d\vec{R} = \lim_{\substack{n\to\infty \\ \max \Delta s_i \to 0}} \sum_{i=1}^{n} \vec{F}(P_i^*) \cdot \hat{T}(P_i^*)\Delta s_i,$$

where $\hat{T}(P)$ is the unit tanget vector for C at P, and P_i^*, Δs_i are defined in the same way as for scalar line integrals. Let us give two basic laws for vector line integrals:

1. Let $\vec{F}$ be a vector field on a finite directed curve C, and let C be divided into curves C_1 and C_2, both having the same direction as C. Then:
$$\int_C \vec{F} \cdot d\vec{R} = \int_{C_1} \vec{F} \cdot d\vec{R} + \int_{C_2} \vec{F} \cdot d\vec{R}.$$
2. Let $\vec{F}$ and $\vec{G}$ be vector fields on a finite, directed curve C. Then:
$$\int_C \left(a\vec{F} + b\vec{G}\right) \cdot d\vec{R} = a\int_C \vec{F} \cdot d\vec{R} + b\int_C \vec{G} \cdot d\vec{R}.$$

There are two ways of evaluating vector line integrals. First, we can evaluate using the definition of the integral as the limit of a Riemann sum. We can also evaluate by parameter. Let $\vec{R}(t)$ be a path for C, with t from a to b, $\vec{R}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. Since $\hat{T} = \frac{d\vec{R}}{dt} \Big/ \left|\frac{d\vec{R}}{dt}\right|$ and $\frac{ds}{dt} = \left|\frac{dR}{dt}\right|$, we get that

$$\int_C \vec{F} \cdot d\vec{R} = \int_C \left(\vec{F} \cdot \hat{T}\right) ds = \int_a^b \left(\vec{F} \cdot \frac{d\vec{R}}{dt}\right) dt$$

2 Conservative Fields

Definition 1 *Let D be a domain for a vector field $\vec{F}$. If for any points $A, B \in D$ and any curve $C \in D$ from A to B, the integral $\int_C \vec{F} \cdot d\vec{R}$ depends only on A and B and not on the choice of C, then $\vec{F}$ is called a* conservative field.

An equivalent definition is the following:

Definition 2 *Let D be a domain for a vector field $\vec{F}$. If for any simple closed curve $C \in D$, $\int_C \vec{F} \cdot d\vec{R} = 0$, then $\vec{F}$ is called a conservative field.*

Any conservative field has a scalar potential.