Problem Set V Solutions

1. $\oint$11.2 2a.

   (a) The values are : $y = 4, x = 2, w = 8$.

   (b) We write $w = f(x, y, z) = xyz^2, x = g(y, z, t) = \frac{yz^2}{t}$, $y = h(z, t) = zt^2$. Then:

   $$dw = f_x dx + f_y dy + f_z dz,$$

   $$dx = g_Z dz + g + y dy + g_t dt,$$

   $$dy = h_z dz + h_t dt.$$

   Hence:

   $$dw = (f_x g_z + f_y h_z + f_z + f_x g_y h_z) dz + (f_x g_t + f_y h_t + f_x g_y h_t) dt.$$

   Since $f_x = yz^2 = 4, f_y = xz^2 = 2, f_z = 2xyz = 16, g_z = \frac{2yz}{t} = 4, g_y = \frac{z^2}{t} = \frac{1}{2}, g_t = -\frac{yz^2}{t^2} = -1, h_z = t^2 = 4, h_t = 2zt = 4$, we get that:

   $$\left(\frac{\partial w}{\partial z}\right)_t = \frac{dw}{dz} = f_x g_z + f_y h_z + f_z + f_x g_y h_z = 48.$$

   (c) In the same manner:

   $$\left(\frac{\partial w}{\partial t}\right)_z = \frac{dw}{dt} = f_x g_t + f_y h_t + f_x g_y h_t = 12.$$

2. $\oint$11.4 14.

   (a) We denote $x = g(u, v, y)$ and consider a new variable $w = f(x, u, v, y) = x^2 - 4uvx + y^2$. Then $dw = f_x dx + f_u du + f_v dv + f_y dy$, and since $dx = g_u du + g_v dv + g_y dy$, it follows that $dw = (f_x g_u + f_u) du + (f_x g_v + f_v) dv + (f_x g_y + f_y) dy$. Since $w = 0$, $\frac{dw}{dv} = 0$, so $g_v = -\frac{f_v}{f_x}$. Hence

   $$\left(\frac{\partial x}{\partial v}\right)_{u,y} = g_v = -\frac{f_v}{f_x} = -\frac{4ux}{2x - 4uv} = \frac{2ux}{x - 2uv}.$$

(b) If $u = v = y = 1$, then $x^2 - 4x + 1 = 0$, so $x = 2 \pm \sqrt{3}$, and

$$(\frac{\partial x}{\partial v})_{u,y} = \frac{2x}{x-2} = 2\frac{2 \pm \sqrt{3}}{\pm\sqrt{3}} = 2 \pm \frac{4}{\sqrt{3}}.$$

3. $\oint$11.6 2.

   (a) Let us denote $P = (3, 1, 2)$. The gradient of f at P is $\vec{\nabla} f|_P = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = 2x\hat{\imath} - 2y\hat{\jmath} + 2z\hat{k} = 6\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$.

   (b) The plane normal to the surface has an equation of the form $6x - 2y + 4z = a$, for some $a \in \mathbb{R}$. Introducing the values of x, y, and z, we get that $a = 24$. So the plane equation is $3x - y + 2z = 12$.

4. $\oint$11.4 11.

   Let $f(x, r, \theta) = x - r\cos\theta = 0$ and $g(y, r, \theta) = y - r\sin\theta = 0$. Then $f_x dx + f_r dr + f_\theta d\theta = 0$ and $g_y dy + g_r dr + g_\theta d\theta = 0$, so

$$dx - \cos\theta dr + \sin\theta r d\theta = 0,$$

$$dy - \sin\theta dr - \cos\theta r d\theta = 0.$$

   If we take $dy = 0$, then $d\theta = -\frac{\sin\theta}{\cos\theta}\frac{1}{r}dr$. Hence $dx = \cos\theta dr + \frac{\sin^2\theta}{\cos\theta}dr = \frac{1}{\cos\theta}dr$. So

$$\left(\frac{\partial r}{\partial x}\right)_y = \frac{dr}{dx} = \cos\theta.$$

   If we take $dx = 0$, then $dr = \frac{\sin\theta r}{\cos\theta\theta}d\theta$ and $dy = \sin\theta dr + \cos\theta r d\theta = \frac{1}{\cos\theta}r d\theta$. So

$$\left(\frac{\partial\theta}{\partial y}\right)_x = \frac{\cos\theta}{r}.$$

5. $\oint$12.1 2.

   The global max points are 0 and 2π with the value of g being 1. The global min point is π with the value of g being -1. There are no other local extreme points. This can be easily deducted from the graph of $g(x) = \cos x$ on $[0, 2\pi]$.

6. $\oint$12.4 1f.

   $\vec{\nabla} f(x, y) = (4x - y - 3)\hat{\imath} + (-x - 6y + 7)\hat{\jmath}$, so the extreme points of f are solutions of the following system of equations: $4x - y - 3 = 0$,

$-x - 6y + 7 = 0$. The unique solution is $x = y = 1$. Clearly f doesn't have global extreme points, since for $x = 0$ and $y \to \infty$, $f \to -\infty$, and for $y = 0$ and $x \to \infty$, $f \to \infty$. Hence $P = (1, 1)$ can only be a local extreme point. Using the two-variable test, $f_{xx}(P)f_{yy}(P) - (f_{xy}(P))^2 = -25 < 0$, so P is a saddle point for f.

7. $\oint$12.4 1h.

$$\vec{\nabla} f(x, y) = \frac{x \sin xy}{\cos^2 xy}\hat{\imath} + \frac{y \sin xy}{\cos^2 xy}\hat{\jmath}$$

The solutions for the equation $\vec{\nabla} f = 0$ are given by $xy = n\pi$, for some $n \in \mathbb{Z}$. For a solution point $P(x, y)$,$xy = n\pi$, if n is even, $f(P) = 1$, and if n is odd, $f(P) = -1$. Considering the fact that there exists a neighborhood U of P such that all points (a, b) in U satisfy $n - \frac{\pi}{2} < ab < n + \frac{\pi}{2}$, so $f(a, b)$ has the same sign as $f(x, y)$, P is a local minimum point if n is even and P is a local maximum point if n is odd. Clearly P cannot be a global extreme point.

8. $\oint$12.2 7.

   (i) The set is closed, since it contain all its boundary points : $y = x$ and $y = x^2$ for all $x \in [0, 1]$. Also, the set is contained in the square with vertices $(0, 0), (0, 1), (1, 0)$, and $(1, 1)$, since $0 \geq x^2 \geq y \geq x \geq 1$. Hence the set is bounded, so it is compact.

   (ii) The set is the graph of the continuous function $f(x) = y = x^2$ in $\mathbf{E^2}$, so it is closed. Since this function is not bounded, the set is not bounded, so it is not compact.

   (iii) The set is open, since the boundary points, given by $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ are not contained in the set. Hence the set is not compact.

   (iv) The set isn't open, since the boundary points given by $x^2+y^2+z^2 = 1$ are contained in the set. It is not closed either, since the boundary points given by $x^2 + y^2 + z^2 = 4$ are not contained in the set. Hence the set is not compact.

9. $\oint$13.2 2.

We can consider $x + 2y + 2z = 12$, since if $x + 2y + 2z = \alpha < 12$, then $x' = \frac{12}{\alpha}x, y' = \frac{12}{\alpha}y$, and $x' = \frac{12}{\alpha}y$ give a bigger volume than x, y, and z. Hence the constraint is $g(x, y, z) = x + 2y + 2z = 12$. Let us compute the constrained critical points P. The equation $\vec{\nabla} f|_P = \lambda^P \vec{\nabla} g|_P$ is equivalent to the system of equations $yz = \lambda^P, xz = 2\lambda^P, xy = 2\lambda^P$ for some λ^P. The solutions of this system of equations are: $x = y = \lambda^P = 0, z \in \mathbb{R}$, $x = z = \lambda^P = 0, y \in \mathbb{R}$, $y = z = \lambda^P = 0, z \in \mathbb{R}$, and $y = z = \frac{x}{2} = \sqrt{\lambda^P}$. Clearly if $x = 0, y = 0$, or $z = 0$ we do not get a maximum volume, so the maximum must be given by $y = z = \frac{x}{2}$. Using $g(x, y, z) = 12$, we get $x = 4$, $y = z = 2$, and the maximum volume is 16.

10. $\oint$13.2 8.

    Let us denote the ratio $\frac{x}{y}$ by t. Since the volume is fixed, $x^2 y = g(x, t) = x^3 t = a$. We want to find t such that the area $x^2 + 4xy = f(x, t) = x^2 + 4x^2 t$ is minimum. The constrained critical points are given by $3x^2 t = \lambda(2x + 8xt)$ and $x^3 = 4\lambda x^2$. Hence $x = 0$ or $x = 4\lambda$. For $x = 0$, $t = \infty$, so $x = 4\lambda$, and $t = \frac{1}{2}$.

11. $\oint$13.2 16.

    The constraint functions are $g_1(x, y, z) = x + y - z = 3$ and $g_2(x, y, z) = x - y + z = 1$. The distance from the origin is given by $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Hence the constrained critical points are satisfy the following equations for some λ_1 and λ_2: $\frac{x}{\sqrt{x^2+y^2+z^2}} = \lambda_1 + \lambda_2$, $\frac{y}{\sqrt{x^2+y^2+z^2}} = \lambda_1 - \lambda_2$, $\frac{z}{\sqrt{x^2+y^2+z^2}} = \lambda_2 - \lambda_1$. From the last two equations we get that $y = -z$, so $x + 2y = 3$, $x - 2y = 1$. Hence $x = 2, y = \frac{1}{2}$, and $z = -\frac{1}{2}$. So the point with minimal distance to the origin is $P(2, \frac{1}{2}, -\frac{1}{2})$.

12. $\oint$13.6 12.

    Since $xy = 1$, $y = \frac{1}{x}$, so the distance to $P(2, -2)$ is given is minimal when $(x - 2)^2 + (y + 2)^2 = f(x) = x^2 - 4x + 8 + \frac{4}{x} + \frac{1}{x^2}$ is minimal. Hence $\frac{df}{dx} = 2x - 4 - \frac{4}{x^2} - \frac{2}{x^3} = \frac{2}{x^3}(x^2 + 1)(x^2 - 1 - 2x) = 0$ has the unique solution $x = 1 + \sqrt{2}$ for $x > 0$. Then $y = \frac{1}{x} = \sqrt{2} - 1$. So the point in the first quadrant on the hyperbola $xy = 1$ closest to $P(2, -2)$ is $Q(1 + \sqrt{2}, \sqrt{2} - 1)$.