Problem Set VIII Solutions

1. $\oint$ 19.3 1b.

   Since $\vec{R}(t) = \cos t\hat{i} + \sin t\hat{j} + \hat{k}$, then

$$\left|\frac{d\vec{R}}{dt}\right| = |\sin t\hat{i} - \cos t\hat{j} + \hat{k}| = \sqrt{2}$$

   Hence

$$\int_C f\, ds = \int_0^{2\pi} \left(\sqrt{2}(t+1)\sin t\cos t\right) dt = \sqrt{2}\int_0^{2\pi} (t\sin t\cos t)\, dt =$$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi} (2t\sin 2t)\, dt = \frac{\sqrt{2}}{8}\int_0^{4\pi} (t\sin t)\, dt =$$

$$= \frac{\sqrt{2}}{8}\left(-(t\cos t)\,|_0^{4\pi} - \frac{1}{8}\int_0^{4\pi} -\cos t dt\right) = \frac{\sqrt{2}\pi}{2}$$

2. $\oint$ 19.3 2b.

$$\vec{F}(t) = (t+\sin t)\hat{i} + (t+\cos t)\hat{j} + (\sin t + \cos t)\hat{k}$$

   and

$$\frac{d\vec{R}}{dt} = \sin t\hat{i} - \cos t\hat{j} + \hat{k}.$$

   Hence

$$\int_C \vec{F}\cdot d\vec{R} = \int_0^{2\pi} \vec{F}\cdot\frac{d\vec{R}}{dt}dt =$$

$$= \int_0^{2\pi} (\sin^2 t + t\sin t - \cos^2 t - t\cos t\sin t + \cos t)dt =$$

$$= \int_0^{2\pi} (-\cos 2t)dt + \int_0^{2\pi} t(\sin t - \cos t)dt =$$

$$= \int_0^{2\pi} t(\sin t - \cos t)dt =$$

$$= -(\sin t + \cos t)t|_0^{2\pi} + \int_0^{2\pi} (\sin t + \cos t)dt = -2\pi$$

3. $\oint$19.5 4.

Let C_1 be the directed line segment from $(0,0,0)$ to $(1,0,0)$, C_2 the directed line segment from $(1,0,0)$ to $(2,1,2)$, and C_3 the directed line segment from $(0,0,0)$ to $(2,1,2)$. Then

$$\int_C \vec{F}\cdot d\vec{R} = \int_{C_1} \vec{F}\cdot d\vec{R} + \int_{C_2} \vec{F}\cdot d\vec{R} - \int_{C_3} \vec{F}\cdot d\vec{R}$$

For C_1, the parametric equations are $x = t, y = 0, z = 0$, as t goes from 0 to 1. Then $\frac{d\vec{R}}{dt} = \hat{\imath}$ and $\vec{F} = t^2\hat{\imath}$. Hence

$$\int_{C_1} \vec{F}\cdot d\vec{R} = \int_0^1 t^2 dt = \frac{1}{3}$$

For C_2, the parametric equations are $x = 1+t, y = t, z = 2t$, as t goes from 0 to 1. Then $\frac{d\vec{R}}{dt} = \hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\vec{F} = (1+t)^2\hat{\imath} + (t^2+t)\hat{\jmath} + 2t\hat{k}$. Hence

$$\int_{C_2} \vec{F}\cdot d\vec{R} = \int_0^1 (2t^2 + 7t + 1)dt = \frac{31}{6}$$

For C_3, the parametric equations are $x = 2t, y = t, z = 2t$, as t goes from 0 to 1. Then $\frac{d\vec{R}}{dt} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$ and $\vec{F} = 4t^2\hat{\imath} + 2t^2\hat{\jmath} + 2t\hat{k}$. Hence

$$\int_{C_3} \vec{F}\cdot d\vec{R} = \int_0^1 (10t^2 + 4t)dt = \frac{16}{3}$$

So the integral on C is

$$\int_C \vec{F}\cdot d\vec{R} = \int_{C_1} \vec{F}\cdot d\vec{R} + \int_{C_2} \vec{F}\cdot d\vec{R} - \int_{C_3} \vec{F}\cdot d\vec{R} = \frac{1}{6}$$

4. $\oint$19.6 1.

We can define C as the set of points (x,y) with $x = a\cos t, y = a\sin t$, with t from 0 to 2π. Then $\vec{F} = r\hat{r} = a\cos t\hat{\imath} + a\sin t\hat{\jmath}$. Hence

$$\int_C \vec{F}ds = \left(\int_0^{2\pi} a^2\cos t dt\right)\hat{\imath} + \left(\int_0^{2\pi} a^2\sin t dt\right)\hat{\jmath} = 0$$

5. $\oint$20.3 1c.

Let $f(x,y,z) = xyz$. Then $f_x = yz, f_y = xz, f_z = xy$, so $\vec{F} = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k} = \vec{\nabla} f$, i.e. f is a scalar potential for $\vec{F}$. Hence

$$\int_P^Q \vec{F}\cdot d\vec{R} = f(Q) - f(P) = 8 - 8 = 0$$

6. $\oint$ 20.3 4b.

   Clearly $\vec{F} = x\hat{i}$ is a gradient field and $f = \frac{x^2}{2}$ is a scalar potential for $\vec{F}$. Then

$$\int_C \vec{F} \cdot d\vec{R} = f(1,0) - f(-1,0) = 0$$

7. $\oint$20.4 2a.

   Let $P = (a,b,c)$ and $\vec{R} = at\hat{i} + bt\hat{j} + ct\hat{k}$, with t from 0 to 1. Then $\vec{F} = (abct^2+abt+act+a)\hat{i}+(abct^2+abt+bct+b)\hat{j}+(abct^2+act+bct+c)\hat{k}$, so

$$\int_0^1 \vec{F} \cdot \frac{d\vec{R}}{dt} dt = \int_0^1 (3abct^2 + 2abt + 2act + 2bct + a + b + c) dt =$$
$$= abc + ab + ac + bc + a + b + c$$

   Now if we put x, y, z instead of a, b, c, we obtain the scalar potential $f(x,y,z) = xyz + xy + xz + yz + x + y + z$. It is easy to verify that f is indeed a scalar potential for $\vec{F}$.

8. $\oint$20.5 1.

   Let f be the scalar potential we want to find. Then

$$f(x,y,z) = \int (ye^{xy} + ze^{xz}) dx + a(y,z) =$$
$$= e^{xy} + e^{xz} + a(y,z).$$

   Hence

$$xe^{xy} + 2 = f_y = xe^{xy} + \frac{\partial a(y,z)}{\partial y}, \qquad so \qquad \frac{\partial a(y,z)}{\partial y} = 2.$$

   Hence

$$a(y,z) = 2y + b(z), \qquad f(x,y,z) = e^{xy} + e^{xz} + 2y + b(z).$$
$$xe^{xz} = f_z = xe^{xz} + b'(z), \qquad so \qquad b(z) = c$$

   We can take $b(z) = 0$, and obtain a scalar potential

$$f(x,y,z) = e^{xy} + e^{xz} + 2y.$$

   The given curve C goes from $P(0,0,1)$ to $Q(1,1,0)$. Hence

$$\int_C \vec{F} \cdot d\vec{R} = f(Q) - f(P) = e + 1$$