Problem Set I Solutions

1. $\oint$1.3 1. for $\mathbf{E^3}$.

   (a) True.

   (b) False.

   (c) False.

   (d) True.

   (e) True.

   For (b) and (c) it is easy to find counter-examples. It follows imediately from properties 1.3.3, 1.3.4, and 1.3.5 that (a), (d), and (e) are true.

2. $\oint$1.4 2. for $\mathbf{E^3}$.

   (a) False.

   (b) True.

   (c) True.

   (d) False.

   For (a) and (d) it is easy to find counter-examples. Sentences (b) and (c) follow shortly from properties 1.4.8 and 1.4.9.

3. $\oint$1.4 3b.

   We can conclude that the sum of the vertex angles is smaller than 2π. Consider the non-planar quadrilateral $P_1P_2P_3P_4$. The planes

$(P_1P_2P_3)$ and $(P_1P_4P_3)$ are distinct. The total sum of the vertex angles in triangles $P_1P_2P_3$ and $P_1P_4P_3$ is 2π. But by the law of cosines, $P_2P_1P_3 + P_4P_1P_3 > P_2P_1P_4$ and $P_2P_3P_1 + P_4P_3P_1 > P_2P_3P_4$. Hence the sum of the vertex angles of the quadrilateral is smaller than the total sum of the angles in the triangles $P_1P_2P_3$ and $P_1P_4P_3$.

4. $\oint$1.5 1.

   Yes, P_1 and P_3 must be the same point. Since PP_2 is perpendicular to the plane M and P_2P_3 is perpendicular to the line L, the plane PP_2P_3 is perpendicular to the line L, hence the line PP_3 is perpendicular to L. So the point P_3 is the projection of P on L, i.e. it is the same point as P_1.

5. $\oint$1.5 2.

   Let M and M' intersect at line L. Let the rectangle be $P_1P_2P_3P_4$, and let its projection on M' be $P_1'P_2'P_3'P_4'$. At least one of the sides of the rectangle intersects line L. Suppose this line is P_1P_2 and denote the intersection by R. Also consider P_1 to be between R and P_2. R,P_1', and P_2' are colinear. Suppose P_1P_4 intersectes line L as well, and let S be their intersection. Then in the tetrahedron $RSP_1'P_1$ we will get that the angles $RP_1'S, RP_1'P_1, SP_1'P_1$ and RP_1S are all right angles. But $RP_1 > RP_1'$ and $SP_1 > SP_1'$, so $RP_1^2 + SP_1^2 > RP_1'^2 + SP_1'^2 = RS^2$. Hence the angle RP_1S is not right. So the supposition is false, P_1P_4 does not intersect the line L. So P_1P_4 and $P_1'P_4'$ are parallel. Then $P_1P_4 = P_1'P_4' = c$. $P_1P_4 \neq 2$, since then the area of the projection is bigger than the area of the rectangle. Hence $P_1P_4 = c = 1$. Then by 1.5.5, we get that $cos\theta = \frac{1}{2}$, where $0ta$ is the acute angle between M and M'. Hence $\theta = \frac{\pi}{3}$.

6. $\oint 1.6$ 6.

Suppose there is an n-gon face with $n \geq 5$. Then each of the n edges of this faces is contained in at another distinct face, so in total we have at least $n+1 \geq 6$ faces. Hence all the faces are triangles or quadrilaterals. Let a be the number of quadrilateral faces. Let us add together the number of edges on all faces. The result will be $4a + 3(5-a) = 15 + a$. Since each edge belongs to exactly two faces(the 5-hedron is convex), we counted each edges twice, i.e. $15 + a = 2E$. Hence a is odd, i.e. the 5-hedron has an odd number of quadrilateral face. So this solves points (a) and (b).

Suppose all the faces are quadrilaterals. Then $a = 5$ and $E = 10$, so by Euler's formula, $V = 7$. But if we add together the number of vertices on all faces, we get $4F$. Since each vertex belongs to at least 3 faces, $4F \geq 3V$, which is clearly false. Hence not all faces are quadrilaterals.

7. $\oint 1.6$ 10.

Let V_F be the number of vertices on each face. Let F_V be the number of faces each vertex belongs to. Adding together all vertices on each face, we get FV_F. But vertex was counted F_V times. Also, it is the same as counting each edge twice, since each edge belongs to two faces and V_F is also the number of edges on each face. So

$$F \cdot V_F = F_V \cdot V = 2E.$$
$$E = F\frac{V_F}{2}$$
$$V = F\frac{V_F}{F_V}$$

Hence

$$F(1 - \frac{V_F}{2} + \frac{V_F}{F_V}) = 2,$$

so

$$(2 - V_F)F_V + 2V_F > 0$$
$$1 + \frac{2}{V_F - 2} > \frac{F_V}{2} \geq \frac{3}{2}.$$

Hence $V_F \leq 5$.

(a) If $V_F = 3$, the following equalities hold:

$$F = \frac{F_V}{3}V = V(\frac{F_V}{2} - 1) + 2$$

Hence

$$V(1 - \frac{F_V}{6}) = 2$$

So $F_V \leq 5$. For the case $F_V = 3$ we get the regular tetrahedron, with 4 triangular faces and 4 vertices. For $F_V = 4$ we get the regular octahedron, with 8 triangular faces and 6 vertices. For $F_V = 4$ we get the regular icosahedron, with 20 triangular faces and 12 vertices.

(b) If $V_F = 4$, the following equalities hold:

$$F = \frac{F_V}{4}V = V(\frac{F_V}{2} - 1) + 2$$

Hence

$$V(1 - \frac{F_V}{4}) = 2$$

But F_V must be greater than 2, so $F_V = 3$. In this case, we obtain the regular hexadron(the cube), with 6 square faces and 8 vertices.

(c) If $V_F = 5$, the following equalities hold:

$$F = \frac{F_V}{5}V = V(\frac{F_V}{2} - 1) + 2$$

Hence

$$V(1-\frac{3F_V}{10})=2$$

But F_V must be greater than 2, so $F_V = 3$. In this case, we obtain the regular dodecahedron, with 12 pentagonal faces and 20 vertices.

8. $\oint$1.6 14.

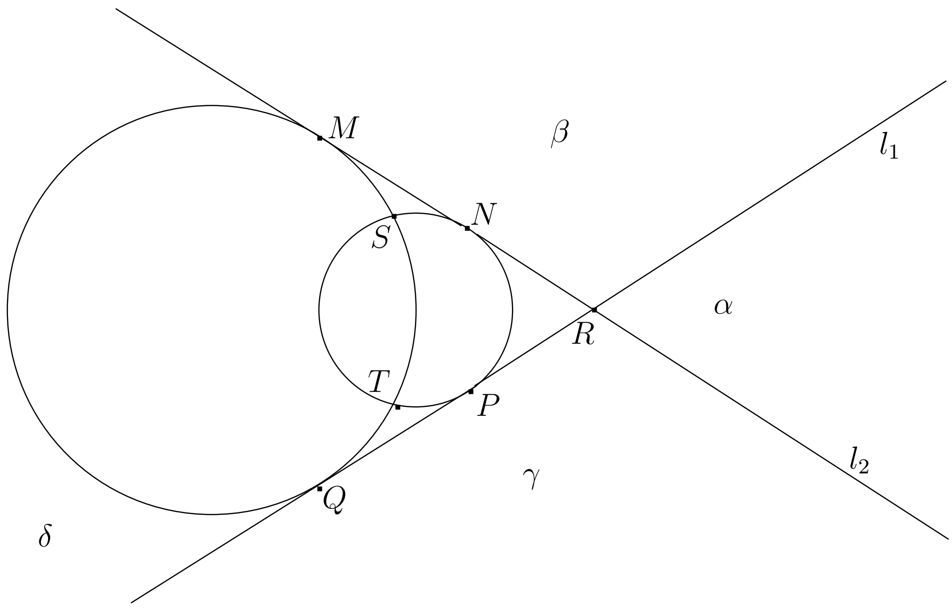

Figure 1: For problem 8

(a) For $n = 0$, the set of points is the union of regions α, δ , the interior of the two circles and the interior of triangle NPR.

(b) For $n = 1$, the set of points is formed by points S, T, M, N, P, Q and the lines l_1 and l_2 minus line segments MN and PQ and point R.

(c) For $n = 2$, the set of points is the union of β, γ, and the arcs MS, NS, QT, and TP, excluding points M, N, P, Q, S, T.

(d) For $n = 3$, the set of points is $MN \cup PQ$, excluding points M, N, P, and Q.

(e) For $n = 4$, the set of points is the interior of the region formed by line segment MN and arcs MS and NS plus the interior of the region formed by line segment PQ and arcs PT and QT.

(f) For $n = \infty$, the set of points is formed by point R.

9. $\oint$2.4 2.

The condition is equivalent to $(a+b+c)\vec{A}+(c-b)\vec{B} = \vec{0}$, i.e. $a+b+c = 0$ and $b = c$. For $a = 2, b = c = -1$ these equalities hold.

10. $\oint$2.4 4.

In clockwise order starting with the side right after $\vec{B}$, the expressions of the sides are: $\vec{B} - \vec{A}$, $-\vec{A}$, $-\vec{B}$, $\vec{A} - \vec{B}$. This can be easily observed using the parallelogram rule of adding and subtracting vectors and the properties of the regular hexagon.

11. $\oint$2.4 10.

We need to find $\vec{C}$ and $\vec{D}$ such that $\vec{B}\times\vec{C} = \vec{0}$, $\vec{B}\cdot\vec{D} = 0$, and$\vec{C}+\vec{D} = \vec{A}$. Since $\vec{B} \times \vec{C} = \vec{0}$, $\vec{C} = a\vec{B}$ for some scalar a. Hence $\vec{D} = \vec{A} - a\vec{B}$. Now from $\vec{B}\cdot\vec{D} = 0$, we get that $\vec{A}\cdot\vec{B} = a\vec{B}\cdot\vec{B}$. So $\vec{C} = a\vec{B}$ and $\vec{D} = \vec{A}-a\vec{B}$, where $a = \frac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}}$.

12. $\oint$2.4 14.

(a) Let P be the projection of R on $\vec{A}$. Let θ be the angle between $\vec{R}$ and $\vec{A}$. then $\vec{R} \cdot \vec{A} = OR|\vec{A}|\cos\theta = c$. Hence $OP = \frac{c}{|\vec{A}|}$ is a constant. So the set of points R with $\vec{R} \cdot \vec{A} = c$ is a plane perpendicular on OA. For c positive this plane intersects the ray $\vec{OA}$, and for c negative it intersects the ray with direction opposite from $\vec{OA}$.

(b) Since $\vec{R} \times \hat{u}$ has the same direction as $\vec{A}$, $\vec{R}$ is in the plane perpendicular on $\vec{A}$ that goes through O. More precisely the half-plane determined by the right-hand rule. $OR \sin\theta = |\vec{A}|$, so the distance from R to the line determined by $\hat{u}$ is constant. This line is the border of the half-plane. Hence the set of all points R such that $\vec{R} \times \hat{u} = \vec{A}$ is a line in the plane perpendicular to $\vec{A}$ that goes through O.

13. $\oint$2.7 6.

Let $ABCD$ be a tetrahedron. It can be viewed as a quadrilateral in space in three ways: with each pair of opposite edges as the pair of diagonals. In each case, the two other pairs are the edges of the quadrilateral. Applying Theorem 2.7.4 to the quadrilateral, we get that the midpoints of its segments form a parallelogram. Hence the diagonals of this parallelogram bisect each other. But these diagonals are the line segments that join the midpoints of opposite edges in the tetrahedron. Applying Theorem 2.7.4. for the other two cases of quadrilateral, we get that each of these line segments meets the others at their midpoints, so all three of them have a common point, which is the midpoint for each of them.