Lecture VI
Calculus of Vector Functions

Recall that $\frac{d\vec{R}}{dt}$ denotes the first-order derivative of $\vec{R}(t)$, and that $\frac{d^2\vec{R}}{dt^2}$ denotes the second-order derivative of $\vec{R}(t)$. We introduce new notations for these functions: $\frac{d\vec{R}}{dt} = \dot{\vec{R}}(t)$ and $\frac{d^2\vec{R}}{dt^2} = \ddot{\vec{R}}(t)$. Let $\vec{R}(t) = a_1(t)\hat{\imath} + a_2(t)\hat{\jmath} + a_3(t)\hat{k}$. Then the following differentiation rules stand:

1. $\dot{\vec{R}}(t) = \dot{a}_1(t)\hat{\imath} + \dot{a}_2(t)\hat{\jmath} + \dot{a}_3(t)\hat{k}$.
2. $\ddot{\vec{R}}(t) = \ddot{a}_1(t)\hat{\imath} + \ddot{a}_2(t)\hat{\jmath} + \ddot{a}_3(t)\hat{k}$.

   Let $\vec{A}(t)$ and $\vec{B}(t)$ be differentiable vector functions, and let $a(t)$ be a differentiable scalar function. The following differentiation rules stand:

3. $\frac{d}{dt}(\vec{A}(t) + \vec{B}(t)) = \dot{\vec{A}} + \dot{\vec{B}}$.
4. $\frac{d}{dt}(a(t)\vec{A}(t)) = \dot{a}(t)\vec{A}(t) + a(t)\dot{\vec{A}}(t)$.
5. $\frac{d}{dt}(\vec{A}(t) \cdot \vec{B}(t)) = \dot{\vec{A}} \cdot \vec{B} + \vec{A} \cdot \dot{\vec{B}}$.
6. $\frac{d}{dt}(\vec{A}(t) \times \vec{B}(t)) = \dot{\vec{A}} \times \vec{B} + \vec{A} \times \dot{\vec{B}}$.

Theorem 1 (Unit-Vector Theorem) *Let $\vec{R}(t)$ be a unit vector function, $\vec{R}(t) = \hat{u}(t)$ for all t. Then:*

1. *The direction of $\dot{\hat{u}}$ is perpendicular to that of $\hat{u}$. The vector formula describing this is $\hat{u}(t) \cdot \dot{\hat{u}}(t) = 0$ for all t.*
2. *$|\dot{\hat{u}}(t)|$ equals the rate of change of the direction of $\hat{u}(t)$ measured in radians per time.*

The proof for the first part of the Unit-Vector Theorem is quite brief. Since $|\hat{u}(t)| = 1$, $\hat{u}(t) \cdot \hat{u}(t) = 1$. Using differentiation rule 5, we get that $2\hat{u}(t) \cdot \dot{\hat{u}}(t) = \frac{d1}{dt} = 0$, i.e. $\dot{\hat{u}}(t)$ is perpendicular to $\hat{u}(t)$.

Definition 1 *Let $\vec{R}(t)$ be a continuous vector function on an interval $[a, b]$. The set C of all points having $\vec{R}(t)$ as position vector for some t in $[a, b]$ is called a* finite curve*. The function $\vec{R}(t)$ is called a* path *for C.*

For a point P on a curve C, the vector $\frac{d\vec{R}}{dt}$ at P is called the velocity vector at P and is denoted $\vec{v}(P)$. Similarly, the vector $\frac{d^2\vec{R}}{dt^2}$ at P is called the acceleration vector at P and is denoted $\vec{a}(P)$. Let $\vec{d} = \frac{d\vec{a}}{dt} = \frac{d^3\vec{R}}{dt^3}$. We will now define two important scalar functions important for the curve C.

Definition 2 *The* curvature k *is a function defined by* $\mathrm{k}(P) = \frac{|\vec{v}(P)\times\vec{a}(P)|}{|\vec{v}(P)|^3}$.

Definition 3 *The* torsion τ *is a function defined by* $\tau(P) = \frac{[\vec{v}(P),\vec{a}(P),\vec{d}(P)]}{|\vec{v}(P)\times\vec{a}(P)|^2}$.

The curve C is uniquely determined by the functions k and τ.