Lecture XV

Iterated Integrals

In this lecture we look at methods to compute multiple integrals, introducing *iterated integrals*. First, let us define the type of regions for which it can be used. These regions are called *simple regions*.

Definition 1 *A region R in $\mathbf{E^2}$ is called* simple *if it has one of the following properties:*

*(i) There exists an interval $[a, b]$ and there exist continuous functions $g_1(x)$ and $g_2(x)$ defined on $[a, b]$ such that $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$ and the region R is the set of points (x, y) such that $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(y)$. A region with this property is called y-*simple.

*(ii) There exists an interval $[c, d]$ and there exist continuous functions $h_1(y)$ and $h_2(y)$ defined on $[c, d]$ such that $h_1(y) \leq h_2(y)$ for all $x \in [c, d]$ and the region R is the set of points (x, y) such that $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(x)$. A region with this property is called x-*simple.

Definition 2 *Let R be an y-simple region in $\mathbf{E^2}$, with $g_1(x)$ and $g_2(x)$ its lower and upper boundary functions, as in Definition 1(i). Let f be a continuous function on R. For each $x \in [a, b]$ we define $h(x)$ by*

$$h(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)dy$$

Then we compute the integral of h on $[a, b]$, $\int_a^b h(x)dx$. This two-step operation is called a double iterated integral. *The order of integration is first y, then x. Alternatively, we can denote the iterated integral by*

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx.$$

In the same manner we can define iterated integrals for x-simple regions. We say then that the order of integration is first x then y.

Let the region R be the set of all points (x, y) with $0 \le x \le 1$, $x^2 \le y \le x$. We define f as the function on R such that $f(x, y) = xy$ for all $(x, y) \in R$. Let us compute the iterated integral of f:

$$\int_0^1 \int_{x^2}^x xydydx = \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2} \right) dx = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}.$$

We can see that R is also x-simple, with $0 \le y \le 1$ and $h_1(y) = y \le x \le \sqrt{y} = h_2(y)$. Hence we can compute the iterated integral of f in the order x, then y.

$$\int_0^1 \int_y^{\sqrt{y}} xydxdy = \int_0^1 \left(\frac{y^2}{2} - \frac{y^3}{2} \right) dy = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

It is not by chance that the two integrals have the same value. In fact this can be proved for any continuous function f on a x, y-simple region R.

Theorem 1 (Fubini's Theorem) *Let R be a x, y-simple region and let f be a continuous function on R. Then*

$$\int\int_R f dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dxdy,$$

where a, b, c, d and the functions g_1, g_2, h_1, h_2 are as in Definition 1.

The notion of iterated integrals can be extended to $\mathbf{E^3}$. Consider a region R in $\mathbf{E^3}$ such that the projection R' of R into the xy plane is y-simple, and there exist continuous functions $h_1(x, y)$ and $h_2(x, y)$ such that R is the set of all points (x, y, z) with $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, and $h_1(x, y) \le z \le h_2(x, y)$, where a, b, g_1, and g_2 are as in Definiton $1(i)$.Then the triple iterated integral of a function f on R in the order z, y, x is

$$\int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)dz \right] dy \right] dx.$$

In $\mathbf{E^3}$, we can use two methods to compute triple integrals.

(i) Method of slices

Let f be a function on a region R. We define R_z to be the intersection of R with the plane parallel to the xy plane, that intersects the z axis at z. If the double integral $\int\int_{R_z} f(x, y, z)dA$ is a continuous function of f, then

$$\int\int\int_R f dV = \int_{c_1}^{c_2} \left[\int\int_{R_z} f(x, y, z)dA \right] dz,$$

where c_1 and c_2 are the minimum and maximum values for z in R.

(ii) Method of rods

Let be a function on a region R and let R' be the projection of R in the xy plane. For each $(x, y) \in R'$ we define $h_!(x, y)$ and $h_2(x, y)$ to be the minimal and maximal value that z takes when $(x, y, z) \in R$. If R coincides wtih the set of all points (x, y, z) with $(x, y) \in R'$ and $h_1(x, y) \leq z \leq h_@(x, y)$, then

$$\int\int\int_R f dV = \int\int_{R'} \left[\int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z) dz \right] dA.$$