Problem Set XII Solutions

1. $\oint$ 29.2 2.

$$\vec{C} = k\vec{A} = (k, -2k, 0, k, 3k), \quad \vec{D} = \vec{B} - \vec{C} = (2-k, 2k, 4, -k, 1-3k)$$

$$\vec{D} \cdot \vec{A} = 5 - 15k = 0, \qquad so \quad k = \frac{1}{3}$$

Hence

$$\vec{C} = (\frac{1}{3}, -\frac{2}{3}, 0, \frac{1}{3}, 1), \qquad \vec{D} = (\frac{5}{3}, \frac{2}{3}, 4, -\frac{1}{3}, 0)$$

2. $\oint$ 29.3 2.

(a) The unit vectors are

$$e_1 = \left(\frac{3}{5}, -\frac{4}{5}, 00\right), \quad e_2 = \left(\frac{4}{5}, \frac{3}{5}, 0, 0\right),$$

$$e_3 = \left(0, 0, \frac{3}{5}, \frac{4}{5}\right), \quad e_4 = \left(0, 0, \frac{4}{5}, -\frac{3}{5}\right)$$

(b)

$$e_1 \cdot e_2 = \frac{12}{25} - \frac{12}{25} = 0, \qquad e_1 \cdot e_3 = 0, \qquad e_1 \cdot e_4 = 0,$$

$$e_2 \cdot e_3 = 0, \qquad e_2 \cdot e_4 = 0, \qquad e_3 \dot{e}_4 = \frac{12}{25} - \frac{12}{25} = 0$$

Hence the unit vectors are mutually orthogonal, i.e. they form a frame.

(c)

$$(1,1,1,0) \cdot e_1 = -\frac{1}{5}, \qquad (1,1,1,0) \cdot e_2 = \frac{7}{5},$$

$$(1,1,1,0) \cdot e_3 = \frac{3}{5}, \qquad (1,1,1,0) \cdot e_4 = \frac{4}{5},$$

Hence

$$(1,1,1,0) = -\frac{1}{5}e_1 + \frac{7}{5}e_2 + \frac{3}{5}e_3 + \frac{4}{5}e_4$$

(d)

$$-\frac{1}{5}e_1+\frac{7}{5}e_2+\frac{3}{5}e_3+\frac{4}{5}e_4=$$

$$=(-\frac{3}{25}+\frac{28}{25},\frac{4}{25}+\frac{21}{25},\frac{9}{35}+\frac{16}{25},\frac{12}{25}-\frac{12}{25})=(1,1,1,0)$$

3. ∮ 30.6 6.

(a)

$$\begin{bmatrix} -1 & 0 & 1 & 0 & -4 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 & 0 & -4 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix} \to$$

$$\to \begin{bmatrix} -1 & 0 & 1 & 0 & -4 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 & 3 \\ 0 & 0 & 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 1 & 0 & -4 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} -1 & 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$x+z=1, \quad y-z=1, \quad w+z=-3, \qquad so$$

$$x=1-z, \quad y=1+z, \quad w=-3-z$$

(b)

$$w-x=-3-z-(1-z)=-4$$

$$x+y=1-z+1+z=2$$

$$y-z=1+z-z=1$$

$$z + w = z - 3 - z = -3$$

So the solution satisfies the given equations.

4. ∮ 30.7 2.

   (a) The rank of A is 2:
   $$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

   (b) The solution of the system is
   $$x_1 = -x_2 - 2x_4, \qquad x_3 = x_4$$

5. ∮ 30.7 3.

   (a) The rank of the matrix is 3, since taking columns 1,2, and 4 we get
   $$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

   (b) The solution of the system is
   $$x_1 = 2 - x_3 - 2x_5, \qquad x_2 = -1 - 3x_3 - x_5, \qquad x_4 = 6 - 2x_5$$

6. ∮ 30.7 4.

   (a) True

   (b) False

   (c) True

   (d) True

   (e) False

7. ∮ 31.3 1a.

   By Laplace expansions:
   $$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= -1 + 1 + 1 = 1$$

By row reduction:

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1$$

8. $\oint$ 31.3 3.

$$\begin{vmatrix} 1 & 1 & a \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 3a - 3,$$

so $r(A) =$ if and only if $a = 1$. If $a \neq 1$, $r(A) = 2$. For $a = 1$, since

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & b \\ 1 & 2 & 1 \end{vmatrix} = 1 - b,$$

so if $r(A) = 2$, then $r(A : D) = 3$ if and only if $b \neq 1$.

(a) The system has a unique solution if $r(A) = 3$, so $a \neq 1$ and b can take any value.

(b) The system has no solutions if $r(A) < r(A : D)$. Hence $a = 1$ and $r(A : D) = 3$, so $b \neq 1$.

(c) The system has more than one solution if $r(A) = r(A : D) < 3$. Hence $a = 1$ and $b = 1$.

9. $\oint$ 31.4 2.

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} =$$

$$= 1 + 1 = 2$$

$$\begin{vmatrix} 4 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 \end{vmatrix} = 4 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -11 & 0 \\ -1 & 1 & 1 \\ 3 & 0 & 1 \end{vmatrix} =$$

$$= 4 + 2 - 3 - 1 = 2$$

$$\begin{vmatrix} 1 & 4 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 4 & 1 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix} =$$

$$= -2 + 3 - 4 + 1 = -2$$

$$\begin{vmatrix} 1 & 0 & 4 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} =$$

$$= 1 + 3 - 2 + 4 = 6$$

$$\begin{vmatrix} 1 & 0 & 1 & 4 \\ 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 4 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} =$$

$$= 2 - 1 - 4 + 3 = 0$$

Hence

$$x = \frac{2}{2} = 1, \qquad y = \frac{-2}{2} = -1, \qquad w = \frac{6}{2} = 3, \qquad z = 0.$$

10. $\oint$ 32.1 1.

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1-1 & -1 & 1+1 \\ -1+2 & -1 & 1+2 \\ -2+1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 1 & -1 & 3 \\ -1 & -2 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1-1+2 & -1-2+1 \\ 1+2 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 3 & 2 \end{bmatrix}$$

11. $\oint$ 32.4 2.

(a)

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1$$

Hence

$$A^{-1} = Adj(A) = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix}$$

$$\alpha_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \qquad \alpha_{21} = -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0,$$

$$\alpha_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, \qquad \alpha_{12} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1,$$

$$\alpha_{22} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad \alpha_{32} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0,$$

$$\alpha_{13} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad \alpha_{23} = -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1,$$

$$\alpha_{33} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1,$$

hence

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(b) Since $AX = D$, $X = A^{-1}D$:

$$X = A^{-1}D = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

12. $\oint$ 32.5 5.

(a)

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$$

(b)

$$A_{-1} = \frac{1}{2}Adj(A) = \frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c)

$$A^{-1}A = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & \frac{1}{2} - \frac{1}{2} & 1 - 1 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} + \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

$$A^{-1}X = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \qquad so \qquad X = A\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+3 & 2 & 2+6 \end{bmatrix} =$$

$$= \begin{bmatrix} 4 & 2 & 8 \end{bmatrix}$$