18.022 Hour Test
October 28, 2005

CLOSED BOOK; NO BOOKS, NOTES, OR CALCULATORS

Name ______ Rec.Instr. ______ Rec.Time ______

Please show all your work on this paper, and identify your answers clearly. Use backs of pages if necessary. Points for each question are as shown (for a total of 100 points). If you have difficulty on a problem, go on to the next.

1. (20) Let $f(x,y,z) = (x^2 + y^2 + z^2)^{1/2}$. Use linear approximation to estimate $f(Q) - f(P)$, the increase in the value of f from $P = (2,2,1)$ to $Q = (2.1, 1.9, 0.8)$.

$$f_x(P) = \left.\frac{\partial f}{\partial x}\right|_P = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)\Big|_P = \frac{1}{2}\left(\frac{1}{3}\right)(4) = \frac{2}{3}$$

$$f_y(P) = \quad = 2/3$$

$$f_z(P) = \quad = 1/3$$

By linear approx'n, $f(Q) - f(P) \approx f_x \Delta x + f_y(\Delta y) + f_z(\Delta z)$

with $\Delta x = 0.1$, $\Delta y = -0.1$, $z = -0.2$

Hence $f(Q) - f(P) \approx \frac{2}{3}(0.1) - \frac{2}{3}(0.1) - \frac{2}{3}(0.1) = -\frac{2}{30} = \boxed{-\frac{1}{15}}$

SCORE

1.

2.

3.

4.

5.

6.

TOTAL:

2. (10) The temperature at each point (x,y,z) in space at time t is given by the function T(x,y,z,t). The path of a moving particle is given by the position vector $\vec{R}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. Use the chain rule to give an expression, in terms of derivatives of the functions T, x, y, and z, for the instantaneous rate per unit time at which the temperature experienced by the particle is increasing.

Temperature experienced by particle at time t =

$$T(x(t), y(t), z(t), t)$$

B Chain Rule, the desired rate is

$$\frac{dT}{dt} = \boxed{\frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial T}{\partial t}}$$

3. (10) Let the equations f(x,y,z) = 0 and x = g(y), where f(x,y,z) and g(y) are given by mathematical formulas, define y implicitly as a function of x and z. Find $\partial y / \partial x$ in terms of the formal derivatives f_x, f_y, f_z, and g_y. (Suggestion: use the elimination method.)

Unintended error: the given equations cannot hold for given functions f and g unless

$$-\frac{f_x}{f_y} = \frac{1}{g_y}$$ since

$$f(x,y,z) = 0 \Rightarrow \left(\frac{\partial y}{\partial x}\right)_z = -\frac{f_x}{f_y}$$

and $x = g(y) \Rightarrow \left(\frac{\partial y}{\partial x}\right) = \frac{1}{g_y}$.

[grading will be discussed in lecture]

4. (20) Consider the scalar field given by $f(x,y) = xy^2$ in the xy plane. If we limit our attention to points on the circle $x^2 + y^2 = 9$, then f has six constrained critical points. Use a Lagrange multiplier to find these points and show them in a figure. Indicate on the figure which points are global maximum points, global minimum points, local maximum but not global maximum points, and local minimum but not global minimum points. (Hint: use the max-min existence theorem.)

Set up:

$$h(x,y,\lambda) = xy^2 - \lambda(x^2+y^2-9) \Rightarrow$$

$$\left.\begin{aligned} h_x &= y^2 - 2\lambda x = 0 \\ h_y &= 2xy - 2\lambda y = 0 \end{aligned}\right\} \Rightarrow (x-\lambda)y = 0$$

$$h_\lambda = -(x^2+y^2-9) = 0 \qquad \Rightarrow y = 0 \text{ or } x = \lambda$$

Then $y = 0 \Rightarrow x = \pm 3 \Rightarrow (0,-3)$ and $(0,3)$ are crit. points.

$y \neq 0 \Rightarrow y^2 = 2\lambda x = 2x^2 \Rightarrow x^2 + 2x^2 = 9 \Rightarrow$

$x^2 = 3 \Rightarrow x = \pm\sqrt{3}$.

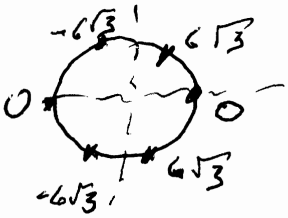
$x = +\sqrt{3} \Rightarrow y^2 = 6 \Rightarrow y = \pm\sqrt{6}$

$x = -\sqrt{3} \qquad \Rightarrow y = \pm\sqrt{6}$

Hence we have 6 crit points:

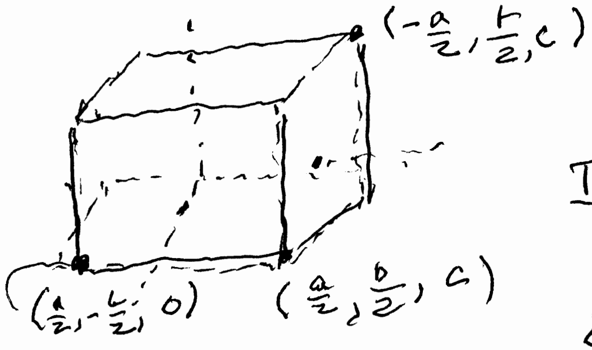
$(3,0)$, $(-3,0)$, $(\sqrt{3},\sqrt{6})$, $(\sqrt{3},-\sqrt{6})$, $(-\sqrt{3},\sqrt{6})$, $(-\sqrt{3},-\sqrt{6})$

and extreme values of f are: 0, 0, $6\sqrt{3}$, $6\sqrt{3}$, $-6\sqrt{3}$, $-6\sqrt{3}$



By existence theorem $6\sqrt{3}$ and $6\sqrt{3}$ points are global max, $-6\sqrt{3}$ and $-6\sqrt{3}$ are global min, Existence theorem applied to {shows $(-3,0)$ is local max and, applied to} shows $(3,0)$ is local min.

5. (20) A solid rectangular block of constant density has total mass M, length = a, width = b, and height = c. Find its moment of inertia about the vertical axis through the centers of the two opposite faces of length a and width b. Express your answer in terms of M, a, b, and (if necessary) c. (The integration is easiest in Cartesian coordinates. Use the z axis as the axis of the moment of inertia.)



δ = density

$$I = \iiint_R \delta(x^2+y^2)\,dV =$$

$$\delta \int_0^c \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)\,dx\,dy\,dz =$$

$$\delta c \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[\frac{x^3}{3} + y^2 x\right]_{-\frac{a}{2}}^{\frac{a}{2}} dy = \delta c \int_{-\frac{b}{2}}^{\frac{b}{2}} \left(\frac{a^3}{12} + y^2 a\right) dy$$

$$= \delta c \left(\frac{a^3}{12} y + \frac{y^3}{3} a\right)\Bigg]_{-\frac{b}{2}}^{\frac{b}{2}} = \delta c \left(\frac{a^3 b}{12} + \frac{b^3 a}{12}\right)$$

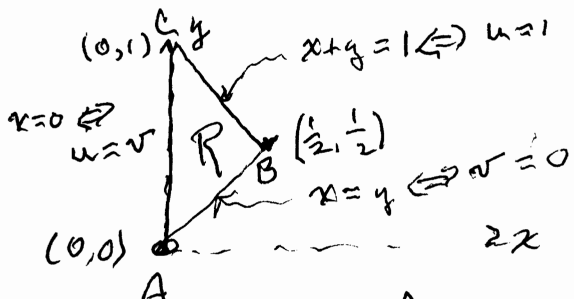
Now $M = \delta abc \Rightarrow \delta = \frac{M}{abc}$

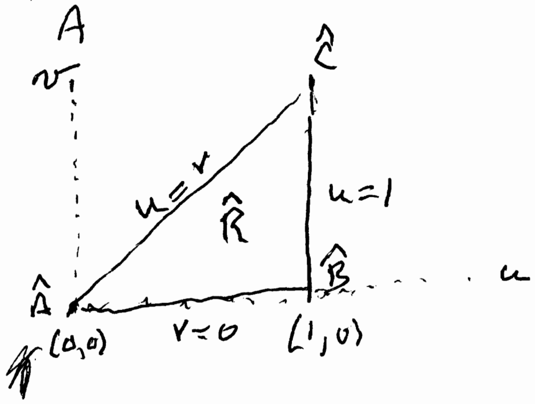
$$\Rightarrow I = \frac{M}{abc} c \left(\frac{a^3 b + b^3 a}{12}\right) = \boxed{\frac{M}{12}(a^2+b^2)}$$

6. (20) You are given the double integral $\iint_R \frac{y-x}{y+x}\,dA$ where R is the triangular region in the xy plane with vertices (0, 0), (1/2, 1/2), and (0, 1).

(a) (15 points) Express this integral as an iterated integral in new variables u and v such that u = y + x and v = y – x. Be sure to show limits of integration.

(b) (5 points) Evaluate the expression found in (a).





Answers: (a) $\int_0^1 \int_0^u \frac{v}{2u}\,dv\,du$

(b) $1/8$

$$\left.\begin{array}{l} u = y+x \\ v = y-x \end{array}\right\} \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \left|\begin{matrix} 1 & 1 \\ -1 & 1 \end{matrix}\right| = 2$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = 1/2$$

Hence $\iint_R = \int_0^1 \int_0^u \frac{v}{u}\left(\frac{1}{2}\right) dv\,du$

$$= \int_0^1 \frac{v^2}{4u}\Big]_0^u du = \int_0^1 \frac{u}{4}\,du = \frac{u^2}{8}\Big]_0^1$$

$$= 1/8$$