CALCULUS OF ONE-VARIABLE SCALAR FUNCTIONS

Definition of *limit.* Let f(x) be a function defined for all x near 0 except possibly at 0. We define the concept: $\lim_{x\to 0} f(x) = L$. (This definition will refer only to values of f(x) for $x \neq 0$. The value of f(x) for x = 0 doesn't matter.) See picture on board, or figures [2-1a] and [2-1b] on p. 93. The definition: there exists a *funnel function* e(t) for 0<t<d for some d (*funnel* means that e(t) decreases steadily to 0 as $t\to 0$) such that $0 < |t| < d \Rightarrow |L-f(t)| < e(t)$.

Derivative of f(x) at x=c (if it exists): Let D(h) = (f(c+h) – f(c)) / h. [D(h) is defined for $h\neq 0$.] Then the derivative df/dx at c is $\lim_{h\to 0} D(h)$, if this limit exists. The *derivative function* f '(x) is defined by f '(c) = df/dx at c as c varies.

Continuity of f(x). Definition: f(x) is *continuous* at x=c if $\lim_{x\to c} f(x) = f(c)$.

Differentiation rules. $(f+g)' = f' + g'$; $(fg)' = f'g + fg'$.

Indefinite integrals. An *indefinite integral* of f(x) is a function F(x) whose derivative is f. We write $\int f(x)dx = F(x) + C$ with arbitrary constant C.

Definite integral. Of f(x) over interval $a \leq x \leq b$. Notation: $\int_a^b f(x)dx$.

Definition: for a given (small) δ, divide interval [a,b] into small non-overlapping *pieces*, each of length < δ. Let n be the number of pieces. Let the piece lengths be $\Delta x_1, \ldots, \Delta x_n$. In each piece, choose an arbitrary *sample point*; let these points be $x_1^*, \ldots, x_n^*$. The quantity $RS_\delta = f(x_1^*)\Delta x_1 + \ldots + f(x_n^*)\Delta x_n$. RS_δ is called a *Riemann sum* of *mesh* δ for f(x) on [a,b]. Consider an infinite sequence of Riemann sums for which the successive meshes approach the limit 0. If, for every such sequence, the RS values in that sequence approach a limit, and if this limit has the same value L for each such sequence, no matter how the pieces and sample points are chosen, we say that the definite integral of f(x) on [a,b] *exists* and *has the value* L.

Basic theorem: *If* f *is a continuous function on* [a,b], *then the definite integral of* f *on* [a,b] *exists.*

Laws for definite integrals. (i) If the integral for f on [a,b] has the value L, then the integral for cf (c a constant) on [a,b] exists and has the value cL. (ii) If the integrals for f and g on [a,b] exist and have the values L_1 and L_2, then the integral for f + g on [a,b] exists and has the value $L_1 + L_2$.

Finding the value of a definite integral. There are two principal direct methods (i) getting an approximate value by using a computer to compute the values of Riemann sums for successively smaller meshes. (ii) finding an indefinite integral F(x) for f(x), in which case the value of the definite integral is F(b) – F(a). This last fact is known as *the fundamental theorem of integral calculus.*

LIMITS AND DERIVATIVES FOR VECTOR FUNCTIONS OF ONE REAL VARIABLE

Let $\vec{R}(t)$ be a vector function of one variable. We define *limit* by using our definition of scalar limit above. We say that $\lim_{t\to c}\vec{R}(t) = \vec{L}$ if and only if $\lim_{t\to c} | \vec{L} - \vec{R}(t) | = 0$.

We define *derivative* by using vector operations in close analogy to our definition of derivative for a scalar function. We let $\vec{D}(h) = (1/h)[\vec{R}(c+h) - \vec{R}(c)]$ and then define $d\vec{R}/dt$ as $\lim_{h\to 0} \vec{D}(h)$. A graphical picture of this derivative is sometimes helpful. Such a picture is given as [3-1] on p. 95. In this picture, we view

Using Cartesian coordinates and an i,j,k representation for a vector function is often helpful in computations. We get $\vec{R}(t) = f(t)\hat{i} + g(t)\hat{k} + h(t)\hat{k}$. It follows from the definition for derivative that $d\vec{R}/dt = f'(t)\hat{i} + g'(t)\hat{j} + h'(t)$.

We define *indefinite integral* in an exactly analogous way. In getting a final general expression for an indefinite integral, the arbitrary constant appears as an arbitrary constant for each coordinate or as a single arbitrary constant *vector.*

For the *definite integral,* we have an analogous definition with the scalar operations in a Riemann sum becoming vector operations of *multiplication by a scalar* and *addition of vectors,* and with the final limit becoming a vector limit. Note that in *evaluating* a definite integral, it will usually be simplest to use the i,j,k representation and evaluate the three resulting scalar integrals.

Differentiation rules for vector operations. See (4.1) through (4.4) on page 98, and (4.9) 0n page 99.

THEOREM ON UNIT VECTOR FUNCTIONS

This theorem appears on pages 100-101. Note that a unit-vector function is a vector function of t where the value of the function is always a unit vector. The theorem tells us that at any point t = c, (i) the vector derivative at c must be orthogonal to the unit vector at c, and (ii) the magnitude of the vector derivative must equal the instantaneous angular rate of change with respect to t (at c) of the direction of the unit vector. This rate is measured in *radians per unit increase in t.*