Lecture XIX

Visualizing Vector Fields; Line Integrals

1 Visualizing Vector Fields

Recall that a vector field in $\mathbf{E^2}$ is a function of the form

$$\vec{F}(x,y) = f_1(x,y)\hat{\imath} + f_2(x,y)\hat{\jmath}.$$

We define two concepts that help us visualize vector fields.

Definition 1 *Let C be a directed smooth curve in $\mathbf{E^2}$, and let $\vec{F}$ be a vector field in $\mathbf{E^2}$. Then C is called an* integral curve of $\vec{F}$ *if, at any point P on C, $\vec{F}(P) \neq 0$ and $\vec{F}(P)$ is tangent to the curve C, i.e. $\vec{F}(P)$ has the same direction as $\hat{T}_P$, the unit tangent vector to C at P.*

Recall that $\mathbf{C}$ is the class of all continuous scalar functions and that $\mathbf{C^1}$ is the class of all differentiable functions with continuous partial derivatives. We say that the vector field $\vec{F}$ as defined above is in $\mathbf{C^1}$ if f_1, f_2 are in $\mathbf{C^1}$.

Lemma 1 *For any point P such that $\vec{F}(P) \neq 0$, there exists an integral curve for F through P.*

Let us take $\vec{F} = x\hat{\imath} + y\hat{\jmath} = r\hat{r}$. The integral curves of F are rays comming out of the origin. $\vec{F}$ passes the derivative test, and it is easy to see that $f = \frac{1}{2}(x^2+y^2)$ is a potential function for f, i.e. $\vec{\nabla} f = x\hat{\imath} + y\hat{\jmath} = r\hat{r}$.

Definition 2 *Let $\vec{F}$ be a gradient field with scalar potential f. The level curves of f, i.e. the curves on which f is constant, are called* equipotential curves for *$\vec{F}$.*

Lemma 2 *Let $\vec{F}$ be a gradient field with scalar potential f. At any point P where $\vec{F}(P) \neq 0$, the integral curve through P is normal to the equipotential curve through P.*

2 Line Integrals

Definition 3 *Let C be a finite curve in $\mathbf{E^2}$ and let f be a scalar field defined on C. We divide C into n pieces of arc-length $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$ and from each piece we choose a point P_i^*, forming the Riemann sum $\sum_{i=1}^{n} f(P_i^*)\Delta s_i$. If the limit*

$$\lim_{\substack{n\to\infty \\ \max \Delta s_i \to 0}} \sum_{i=1}^{n} f(P_i^*)\Delta s_i$$

exists, then it is called the scalar line integral of f on C. *It is denoted by $\int_C f ds$.*

Remember that s is the arc length of C and that $\frac{ds}{dt} = |\frac{d\vec{R}}{dt}|$. We can evaluate line integrals by two methods.

1. Evaluation by definition

   In this method, we simply use the definition of the integral. For example, let C be the circle of radius 2 with center at the origin, and let $f(x, y) = x^2 + y^2$. Then

   $$\int_C f ds = \int_C (x^2 + y^2) ds = \int_C 4 ds = 4 \cdot 4\pi = 16\pi,$$

   since the arc length of C is 4π.

   Let us extend the notion of line integral to vector fields. If $\vec{F}(x, y) = f_1(x, y)\hat{\imath} + f_2(x, y)\hat{\jmath}$, the line integral of $\vec{F}$ on a curve C is

   $$\int_C \vec{F} ds = \int_C f_1 ds \hat{\imath} + \int_C f_2 ds \hat{\jmath}.$$

   Let us take $\vec{F} = x\hat{\imath} + y\hat{\jmath}$ on the circle C of equation $x^2 + y^2 = 4$. Since C is symmetrical with respect to $\vec{Ox}$ and $\vec{Oy}$,

   $$\int_C \vec{F} ds = \int_C -\vec{F} ds, \quad so \quad \int_C \vec{F} ds = 0.$$

2. Parametric evaluation

   Let us take for the finite curve C a path $\vec{R}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$, with t going from a to b. Then

   $$\frac{d\vec{R}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}, \quad so \quad \frac{ds}{dt} = \left|\frac{d\vec{R}}{dt}\right| = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

Hence if $f(x, y)$ is a function on C, we can evaluate the line integral of f:

$$\int_C f ds = \int_a^b f \left| \frac{d\vec{R}}{dt} \right| dt = \int_a^b f(x(t), y(t)) \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt.$$