Problem Set II Solutions

1. $\oint 3.3$ 6.

   Let O be the origin, so that $\vec{A}, \vec{B}$, and $\vec{C}$ can be viewed as position vectors for points A, B, and C.

   (a) By $(\vec{A} \times \vec{B}) \times \vec{C} = \vec{0}$, $\vec{C}$ is parralel to $\vec{A} \times \vec{B}$, so $\vec{C}$ is perpendicular on $\vec{A}$. Hence $\vec{C} \cdot \vec{A} = 0$.

   (b) By $\vec{A} \cdot \vec{A} = \vec{B} \cdot \vec{B} = 4$, $|\vec{A}| = |\vec{B}| = 2$. Since $\vec{A} \cdot \vec{B} = 0$, $\vec{A}$ is perpendicular on $\vec{B}$, so $A \times B = |\vec{A}||\vec{B}| = 4$. And since $\vec{A} \times \vec{B}$ and $\vec{C}$ are parralel, $|\vec{C}| = \frac{8}{|\vec{A} \times \vec{B}|} = 2$.

   (c) Because $(\vec{A} \times \vec{B}) \cdot \vec{C} > 0$, $\vec{C}$ and $\vec{A} \times \vec{B}$ have the same direction. We can choose a Cartesian system such that $\vec{A} = 2\hat{\imath}$ and $\vec{B} = 2\hat{\jmath}$. Then $\vec{C} = 2\hat{k}$, so $\vec{C} \times \vec{B} = -2\vec{A}$

2. $\oint 3.3$ 8.

   Denote the vertices of the triangle by $A(13, 2), B(1, 0, 4)$, and $C(0, 3, 4)$. The area of the triangle is $\frac{AB \cdot AC \cdot \sin B\hat{A}C}{2} = \frac{|\vec{AB} \times \vec{AC}|}{2}$. $\vec{AB} \times \vec{AC} = (-6\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$, so $|\vec{AB} \times \vec{AC}| = 7$. So the area of the triangle is $\frac{7}{2}$.

3. $\oint 3.5$ 6.

   By problem 4, if vertex A is at $(0, 0, 0)$ and vertex B is at $(1, 0, 0)$, then C is at $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and D is at $(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3})$. Let θ be the angle we are looking for. Then $\sin\theta = \frac{\sqrt{6}}{3}$, so $\theta = 54.7$.

4. $\oint 3.6$ 8ac.

   (a)
   $$(\vec{A} \times \vec{C}) \times (\vec{B} \times \vec{C}) = ((\vec{B} \times \vec{C}) \cdot \vec{A})\vec{C} - ((\vec{B} \times \vec{C}) \cdot \vec{C})\vec{A}$$

   But $(\vec{B} \times \vec{C}) \cdot \vec{C} = \vec{B} \cdot (\vec{C} \times \vec{C}) = 0$, so

   $$(\vec{A} \times \vec{B}) \cdot ((\vec{A} \times \vec{C}) \times (\vec{B} \times \vec{C})) = ((\vec{B} \times \vec{C}) \cdot \vec{A})((\vec{A} \times \vec{B}) \cdot \vec{C}) = [\vec{A}, \vec{B}, \vec{C}]^2$$

(c)

$$(\vec{A}\times\vec{B})\times(\vec{A}\times(\vec{B}\times\vec{C})) = ((\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C}))\vec{A}-((\vec{A}\times\vec{B})\cdot A)(\vec{B}\times\vec{C})$$

But $(\vec{A}\times\vec{B})\cdot\vec{A} = \vec{A}\cdot(\vec{A}\times\vec{B}) = (\vec{A}\times\vec{A})\cdot\vec{B} = 0$, so

$$(\vec{A}\times\vec{B})\times(\vec{A}\times(\vec{B}\times\vec{C})) = ((\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C}))\vec{A}$$

Hence

$$\vec{A}\cdot((\vec{A}\times\vec{B})\times(\vec{A}\times(\vec{B}\times\vec{C}))) = ((\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C}))|\vec{A}|^2 =$$

$$((\vec{A}\cdot\vec{B})(\vec{B}\cdot\vec{C})-(\vec{A}\cdot\vec{C})|\vec{B}|^2)|\vec{A}|^2$$

5. $\oint$3.6 12.

   $\vec{Y} = \vec{A}-2\vec{X}$, so $(\vec{X}\times\vec{A})-(2\vec{X}\times\vec{X}) = \vec{B}$, i.e. $\vec{X}\times\vec{A} = \vec{B}$. Then $\vec{B}\times\vec{A} = (\vec{X}\times\vec{A})\vec{A} = (\vec{A}\cdot\vec{X})\vec{A}-(\vec{A}\cdot\vec{A})\vec{X}$. Hence $\vec{X} = \frac{d\vec{A}-\vec{B}\times\vec{A}}{|\vec{A}|^2}$, and $\vec{Y} = \vec{A}-2\vec{X}$.

6. $\oint$4.3 6.

   The distance d between the planes is equal to $\frac{19-5}{|\vec{N}|}$, where $\vec{N} = \hat{\imath}+2\hat{\jmath}+3\hat{k}$. Hence $|\vec{N}| = \sqrt{14}$, so $d = \frac{14}{\sqrt{14}} = \sqrt{14}$.

7. $\oint$4.3 9.

   Since L is defined by the equations $x+2z=1$ and $y=0$, each plane that contains L is given by an equation $x+dy+2z=1$ for some d. Let M_3 be the plane that bisects the acute dihedral angle between M_1 and M_2, and contains L. Then M_3 makes equal acute dihedral angles with M_1 and M_2. Let us consider the vectors N_1, N_2, and N_3 normal to M_1, M_2, and M_3, respectively. Then $\frac{\vec{N_1}\cdot\vec{N_3}}{|\vec{N_1}||\vec{N_3}|} = \frac{\vec{N_2}\cdot\vec{N_3}}{|\vec{N_2}||\vec{N_3}|}$. Hence $\frac{3-2d+12}{7\sqrt{5+d^2}} = \frac{1+2d+4}{3\sqrt{5+d^2}}$, so $d = \frac{1}{2}$. Hence the plan is given by the equation $x+\frac{1}{2}y+2z=1$.

8. $\oint$4.5 2.

   The parametric equations for the line passing through $(2,3,1)$ and $(-1,1,2)$ are $x=-1+3t, y=1+2t, z=2-t$. The equation for the xy plane is $z=0$. So $t=2$, hence the intersection point is $(5,5,0)$.

9. $\oint$4.5 4.

   The vector $\vec{N} = 2\hat{\imath} + 5\hat{\jmath} - \hat{k}$ is normal to the given plane. Then the line defined by $x = 1 + 2t, y = 1 + 5t, z = 1 - t$ contains the point $(1, 1, 1)$ and is perpendicular on the plane, so it contains the projection of $(1, 1, 1)$. So $6 + 30t = 11$, i.e. $t = \frac{1}{6}$. If we enter this value of t in the parametric equations, we get that the projection point is $(\frac{4}{3}, \frac{11}{6}, \frac{5}{6})$.

10. $\oint$4.5 8.

    It can be easily checked that the point with position vector $\frac{d\vec{N}}{|\vec{N}|^2}$ is in the plane M. Then, proceding as in example 5, chapter 4.3, we get that the distance from R to M is $\left|\frac{(\vec{R}_P - \frac{d\vec{N}}{|\vec{N}|^2})\cdot\vec{N}}{|\vec{N}|}\right|$.

11. $\oint$5.4 4.

$$\frac{d}{dt}((\vec{R} \times \frac{d\vec{R}}{dt}) \times \frac{d^2\vec{R}}{dt^2}) =$$
$$= (\frac{d}{dt}(\vec{R} \times \frac{d\vec{R}}{dt})) \times \frac{d^2\vec{R}}{dt^2} + (\vec{R} \times \frac{d\vec{R}}{dt}) \times \frac{d^3\vec{R}}{dt^3} =$$
$$= (\vec{R} \times \frac{d^2\vec{R}}{dt^2}) \times \frac{d^2\vec{R}}{dt^2} + (\vec{R} \times \frac{d\vec{R}}{dt}) \times \frac{d^3\vec{R}}{dt^3}$$

12. $\oint$5.7 1b.

    By integrating $\vec{a}(t)\cdot\hat{\imath}, \vec{a}(t)\cdot\hat{\jmath}$, and $\vec{a}(t)\cdot\hat{k}$, we get that $\vec{v}(t) = (2 - 2\sin t)\hat{\imath} + (2\cos t - 2)\hat{\jmath} + \hat{k}$. By integrating $\vec{v}(t)\cdot\hat{\imath}, \vec{v}(t)\cdot\hat{\jmath}$, and $\vec{v}(t)\cdot\hat{k}$, we get that $\vec{R}(t) = (2t + 2\cos t - 2)\hat{\imath} + (2\sin t - 2t)\hat{\jmath} + t\hat{k}$.

13. $\oint$5.8 1b.

    By integrating $\vec{v}(t)\cdot\hat{\imath}, \vec{v}(t)\cdot\hat{\jmath}$, and $\vec{v}(t)\cdot\hat{k}$, we get that $\vec{R}(t) = (2\cos t + c_1)\hat{\imath} + (2\sin t + c_2)\hat{\jmath} + (3t + c_3)\hat{k}$, for some c_1, c_2, c_3. Then the displacement is $\vec{R}(\frac{\pi}{2}) - \vec{R}(0) = -2\hat{\imath} + 2\hat{\jmath} + \frac{3\pi}{2}\hat{k}$.

14. $\oint$4.5 2.

    Let α be the angle between PB and the line L, where B is the point of position vector $\vec{B}$. Then the distance from B to the projection point of P on L is $|\vec{P} - \vec{B}|\cos\alpha = \frac{(\vec{P}-\vec{B})\cdot\vec{A}}{|\vec{A}|}$. Hence the projecton of P on L is the point of vector position $\frac{(\vec{P}-\vec{B})\cdot\vec{A}}{|\vec{A}|}\vec{A} + \vec{B}$.

15. $\oint$5.2 1d.

    Since $lim_{x \to c} g(x) = l$, there exists ϵ_1 and d_1 such that if $|x-c| \leq d_1$ then $|g(x) - l| < \epsilon_1 |x-c|$. Also, since $lim_{x \to l} h(x) = m$, there exists ϵ_2 and d_2 such that if $|x-l| \leq d_2$ then $|h(x) - m| < \epsilon_2 |x-l|$. Let d be the minimum of d_1 and $\frac{d_2}{\epsilon_1}$ and let $\epsilon = \epsilon_1 \epsilon_2$. Then if $|x-c| \leq d$, $|x-c| \leq d_1$, so $|g(x) - l < \epsilon_1 |x-c| \leq d_2$, so $|h(g(x)) - m| < \epsilon_2 |g(x) - l| \leq \epsilon_2 \epsilon_1 |x-c| = \epsilon |x-c|$. Hence $\epsilon(x)$ is a funnel function for $h(g(x))$, so $lim_{x \to c} h(g(x)) = m$.

16. $\oint$2.7 4.

    It suffices to prove that $\frac{\sin\gamma}{c} = \frac{\sin\beta}{b}$. Let the vertices be P, Q, and R and let $\vec{A} = \vec{RQ}, \vec{B} = \vec{RP}$. Then $a = |\vec{A}|, b = |\vec{B}|$, and $c = |\vec{A} - \vec{B}|$. $\frac{\sin\gamma}{c} = \frac{\sin\beta}{b}$ if and only if $ab\sin\gamma = ac\sin\beta$, i.e. $|\vec{A} \times \vec{B}| = |vecA \times (\vec{A} - \vec{B})|$. But $\vec{A} \times \vec{A} = 0$, so $\vec{A} \times \vec{B} = -\vec{A} \times (\vec{A} - \vec{B})$, so $|\vec{A} \times \vec{B}| = |vecA \times (\vec{A} - \vec{B})|$. Hence $\frac{\sin\gamma}{c} = \frac{\sin\beta}{b}$.