Problem Set IX Solutions

1. $\oint$ 21.1 4.

   (a) Since $r = \sqrt{u^2+v^2}$, $0 \leq r \leq 2$. Clearly, $0 \leq \theta \leq 2\pi$.Since $u = r\cos\theta$ and $v = r\sin\theta$, it follows that

   $$\vec{R}(u,v) = \vec{R}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j} + 2r^2\hat{k}$$

   (b) We take $z = 2(u^2+v^2)$, so $0 \leq z \leq 8$, $0 \leq \theta \leq 2\pi$, and $u = \sqrt{\frac{z}{2}}\cos\theta$, $v = \sqrt{\frac{z}{2}}\sin\theta$. Hence we have that

   $$\vec{R}(\theta,z) = \sqrt{\frac{z}{2}}\cos\theta\hat{i} + \sqrt{\frac{z}{2}}\sin\theta\hat{j} + z\hat{k}$$

2. $\oint$ 21.1 6.

   (a) The domain $\hat{D}$ is the disk of radius a centerd at the origin, and the parametrization is

   $$\vec{R}(x,y) = x\hat{i} + y\hat{j} + \sqrt{a^2-x^2-y^2}\hat{k}$$

   (b) The domain is the rectanle given by $0 \leq r \leq a$ and $0 \leq \theta \leq 2\pi$. The parametrization is

   $$\vec{R}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j} + \sqrt{a^2-r^2}\hat{k}$$

3. $\oint$ 21.3 1ab.

   (a) The parametrization is $\vec{R}(u,v) = u\hat{i}+v\hat{j}+v\hat{k}$, where the domain $\hat{D}$ is the triangle of vertices (0,0), (1,0), and (1,1). Hence the parametric normal vector is

   $$\vec{w}(u,v) = \frac{\partial\vec{R}}{\partial u} \times \frac{\partial\vec{R}}{\partial v} = \hat{i} \times (\hat{j}+\hat{k}) = -\hat{j}+\hat{k}$$

(b) The parametrization is $\vec{R}(\theta, z) = \frac{z}{2}\cos\theta\hat{\imath} + \frac{z}{2}\sin\theta\hat{\jmath} + z\hat{k}$, with $0 \leq z \leq 6$ and $0 \leq \theta \leq 2\pi$. Hence

$$\frac{\partial\vec{R}}{\partial\theta} = -\frac{z}{2}\sin\theta\hat{\imath} + \frac{z}{2}\cos\theta\hat{\jmath},$$
$$\frac{\partial\vec{R}}{\partial z} = \frac{\cos\theta}{2}\hat{\imath} + \frac{\sin\theta}{2}\hat{\jmath} + \hat{k}$$

Hence the parametric normal vector is

$$\vec{w}(\theta, z) = \frac{\partial\vec{R}}{\partial\theta} \times \frac{\partial\vec{R}}{\partial z} = \frac{z}{4}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath} - \hat{k})$$

4. $\oint$ 21.4 1dg.

(d) The parametrization is $\vec{R}(\theta, z) = \sqrt{z^2+1}\cos\theta\hat{\imath} + \sqrt{z^2+1}\sin\theta\hat{\jmath} + z\hat{k}$, with $0 \leq \theta \leq 2\pi$ and $-2 \leq z \leq 2$. Then

$$\frac{\partial\vec{R}}{\partial\theta} = -\sqrt{z^2+1}\sin\theta\hat{\imath} + \sqrt{z^2+1}\cos\theta\hat{\jmath},$$
$$\frac{\partial\vec{R}}{\partial z} = \frac{z}{\sqrt{z^2+1}}\cos\theta\hat{\imath} + \frac{z}{\sqrt{z^2+1}}\sin\theta\hat{\jmath} + \hat{k}$$

Hence the parametric normal vector is

$$\vec{w}(\theta, z) = \frac{z}{\sqrt{z^2+1}}\cos\theta\hat{\imath} + \frac{z}{\sqrt{z^2+1}}\sin\theta\hat{\imath} - z\hat{k}$$

So the surface area is given by

$$\int_{-2}^{2}\int_{0}^{2\pi} |\vec{w}(\theta, z) d\theta dz = \int_{-2}^{2}\int_{0}^{2\pi} \sqrt{2z^2+1} d\theta dz =$$
$$= 4\pi\int_{0}^{2} \sqrt{2z^2+1} dz = 12\pi + \sqrt{2}\pi\ln(3+2\sqrt{2})$$

(g) The parametrization is $\vec{R}(\varphi, \theta) = a\sin\varphi\cos\theta\hat{\imath} + b\ sin\varphi\sin\theta\hat{\jmath} + c\cos\varphi\hat{k}$, with $0 \leq \varphi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$. Hence

$$\frac{\partial R}{\partial\varphi} = a\cos\varphi\cos\theta\hat{\imath} + b\cos\varphi\sin\theta\hat{\jmath} - c\sin\varphi\hat{k},$$
$$\frac{\partial R}{\partial\theta} = -a\sin\varphi\sin\theta\hat{\imath} + b\sin\varphi\cos\theta\hat{\jmath}$$

So $\vec{w}(u, v) = bc\sin^2\varphi\cos\theta\hat{\imath} + ac\sin^2\varphi\sin\theta\hat{\jmath} + ab\sin\varphi\cos\varphi\hat{k}$. The surface area is given by the integral

$$\int_{0}^{\frac{\pi}{2}}\int_{0}^{2\pi} abc\sin\varphi\sqrt{\frac{\sin^2\varphi\cos^2\theta}{a^2} + \frac{a^2c^2\sin^2\varphi\sin^2\theta}{b^2} + \frac{\cos^2\varphi}{c^2}} d\theta d\varphi$$

5. $\oint$ 21.7 1.

   In both cases, C is a loop, i.e. a simple closed curve. For (b) this is obvious, for (a) it is easily verifiable. To obtain a two-sided surface, choose a point P on C and start moving points $P'(t)$ and $P''(t)$ on C away from P in opposite directions. Because C is a loop, P' and P'' will meet again at a point distinct from P. The union of all segments $P'(t)P''(t)$ is a two-sided surface.

6. $\oint$ 22.3 2. Because of symmetry between x and y,

$$\int\int_S x^2 z d\sigma = \int\int_S y^2 z d\sigma = \frac{1}{2}\int\int_S (x^2+y^2) z d\sigma$$

   Let S_1 be the top of the cylinder and let S_2 be the vertical part of S. Then

$$\frac{1}{2}\int\int_S (x^2+y^2) z d\sigma = \frac{1}{2}\int\int_{S_1} r^2 d\sigma + \frac{1}{2}\int\int_{S_2} z d\sigma$$

   Let $t = z - \frac{1}{2}$ and consider the cylinder S' in x,y,t coordinates with function $f(x,y,t) = t$. For $P(x,y,t)$ on S', $P'(x,y,-t)$ is on S' as well, and $f(P) = -f(P')$. Hence $\int\int_{S'} f d\sigma = 0$, so $\int\int_{S_2} z d\sigma = \frac{1}{2}\int\int_{S_2} d\sigma$ If we divide S_2 into n equal cylinders, the Riemann Sum of $\int\int_{S_2} d\sigma$ corresponding to this division is 2π. Hence $\int\int_{S_2} d\sigma = 2\pi$, so

$$\frac{1}{2}\int\int_{S_2} z d\sigma = \frac{1}{4}\int\int_{S_2} d\sigma = \frac{\pi}{2}$$

   So the integral we are evaluating is

$$\int\int_S x^2 z d\sigma = \frac{1}{2}\int\int_{S_1} r^2 d\sigma + \frac{\pi}{2} =$$

$$= \pi\int_0^1 r^3 dr + \frac{\pi}{2} = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

7. $\oint$ 22.3 3.

   (a) As in the previous problem, if we shift the cylinder verticaly by half its altitude, we get that $\int\int_{S'} z d\sigma = 0$, by symmetry with respect to the xy plane. Hence

$$\int\int_S z d\sigma = \frac{3}{2}\int\int_S d\sigma = \frac{3}{2}(2\pi\cdot 2\cdot 3 + 2\pi\cdot 2^2) = 30\pi,$$

   since by looking at the Riemann sum we can see that $\int\int_S d\sigma$ is equal to the area of the closed cylinder.

(b) Because the cylinder is symmetrical with respect to the x axis,

$$\int\int_S x d\sigma = 0$$

(c) Because the cylinder is symmetrical with respect to the yz plane,

$$\int\int_S xyz d\sigma = 0$$

(d)

$$\int\int_S x^2 d\sigma = \int\int_S y^2 d\sigma = \frac{1}{2}\int\int_s (x^2+y^2) d\sigma$$

Let S_1 be the bottom of the cylinder and let S_2 be the vertical part of S. Then

$$\frac{1}{2}\int\int_S (x^2+y^2) d\sigma = \int\int_{S_1} r^2 d\sigma + \frac{1}{2}\int\int_{S_2} 4 d\sigma =$$

$$= 2\pi \int_0^2 r^3 dr + 2(2\pi \cdot 2 \cdot 3) = 32\pi$$

8. $\oint$ 22.5 3abc.

Let S_1 be the bottom of S, let S_2 be the vertical part of S, and let S_3 be the top of S.

(a)

$$\int\int_S (z\hat{k}) \cdot d\vec{\sigma} = 3\int\int_{S_3} (\hat{k} \cdot \hat{k}) dr d\theta + \int\int_{S_2} (z\hat{k} \cdot 2\hat{r}) d\theta dz =$$

$$= 3\int\int_{S_3} dr d\theta = 12\pi$$

(b)

$$\int\int_S (x\hat{\imath} + y\hat{\jmath}) \cdot d\vec{\sigma} = \int\int_S (r\hat{r}) \cdot d\vec{\sigma} =$$

$$= \int\int_{S_1} (-r\hat{r} \cdot \hat{k}) dr d\theta + \int\int_{S_3} (r\hat{r} \cdot \hat{k}) dr d\theta + \int\int_{S_2} (2\hat{r} \cdot 2\hat{r}) d\theta dz =$$

$$= 4\int\int_{S_2} d\theta dz = 48\pi$$

(c)

$$\int\int_S (-y\hat{\imath} + x\hat{\jmath}) \cdot d\vec{\sigma} = \int\int_S (r\hat{\theta}) \cdot d\vec{\sigma} =$$

$$= \int\int_{S_1} (-r\hat{\theta} \cdot \hat{k}) dr d\theta + \int\int_{S_3} (r\hat{\theta} \cdot \hat{k}) dr d\theta + \int\int_{S_2} (2\hat{\theta} \cdot 2\hat{r}) d\theta dz = 0$$

9. $\oint$ 22.5 3de.

Let S_1 be the bottom of S, let S_2 be the vertical part of S, and let S_3 be the top of S.

(d)

$$\int_S (x^2\hat{i} \cdot d\vec{\sigma} =$$

$$= \int\int_{S_1} (x^2\hat{i}) \cdot \hat{k} dr d\theta + \int\int_{S_3} (-x\hat{i} \cdot \hat{k}) dr d\theta + \int\int_{S_2} (x^2\hat{i} \cdot 2\hat{r}) d\theta dz =$$

$$= \int\int_{S_2} x^3 d\theta dz = 0,$$

since S_2 is symmetrical with respect to the x axis, and $(-x)^3 = -x^3$.

(e) In the same manner as in (d), we get

$$\int\int_S (x^3\hat{i}) \cdot d\vec{\sigma} = \int\int_S^2 x^4 d\theta dz = 48 \int_0^{2\pi} \cos^4 \theta d\theta =$$

$$= 24 \int_0^{2\pi} (1 - 2\cos^2\theta \sin^2\theta) d\theta = 48\pi - 12 \int_0^{2\pi} \sin^2(2\theta) d\theta =$$

$$= 48\pi - 6 \int_0^{4\pi} sin^2\theta d\theta = 48\pi - 3 \int_0^{4\pi} d\theta = 36\pi$$

10. $\oint$ 22.6 3.

Since $d\vec{\sigma} = \hat{n} d\sigma$, if we denote $\vec{F} = f\hat{n}$, it follows that

$$\int\int_S f d\vec{\sigma} = \int\int_S f\hat{n} d\sigma = \int\int_S \vec{F} d\sigma$$

This last integral can be evaluated by definition with the help of Riemann sums.

11. $\oint$ 23.1 2.

(a) Not elementary, not regular.

(b) Elementary, therefore regular.

(c) Regular, not elementary.

(d) Regular, not elementary.

12. $\oint$ 23.2 1.

(a) The perimetere of R is not a measure. Suppose we divide R in regular subregions R_1 and R_2. Then if R_1 and R_2 have a common boundary of positive length, this length is equal to $\mu(R_1) + \mu(R_2) - \mu(R)$, so $\mu(R) \neq \mu(R_1) + \mu(R_2)$.

(b) $\mu(R) = 5$ area(R) is a measure. If regular R_1 and R_2 form a subvidision of R, then $R_1 \cap R_2$ doesn't have positive area, so $\mu(R) = \mu(R_1) + \mu(R_2)$. $\mu(R)$ is also an integral measure:

$$\mu(R) = \int\int_R 5dA$$

(c) $\mu(R) = [\text{area}(R)]^2$ is not a measure. Since area$(R) =$ area$(R_1)+$ area(R_2), if area$(R_1) \neq 0$ and area$(R_2) \neq 0$, then $\mu(R) \neq \mu(R_1) + \mu(R_2)$.

(d) $\mu(R) =$ area$(R \cap H_1)-$ area$(R \cap H_2)$ is a measure. Let f be a map f on $\mathbf{E^2}$, such that $f(P) = 1$ for $P \in H_1 - H_2$, $f(P) = 0$ for $P \in H_1 \cap H_2$, and $f(P) = -1$ for $P \in H_2 - H_1$. Then $\mu(R) = \int\int_R f dA$. Hence $\mu(R)$ is an integral measure.

13. $\oint$ 23.2 2.

(a) $\mu(R) = -2$ volume(R) is a measure. In fact, it is an integral measure: $\mu(R) = \int\int\int_R -2dV$.

(b) $\mu(R) =$ volume$(R \cap H_1)-$ volume$(R \cap H_2)$ is a measure. It is an integral measure. We define map f such that $f(P) = 1$ for $P \in H_1 - H_2$, $f(P) = 0$ for $P \in H_1 \cap H_2$, and $f(P) = -1$ for $P \in H_2 - H_1$. Then $\mu(R) = \int\int\int_R f dV$.

(c) $\mu(R) =$ volume$(R \cap T)-$ volume$(R \cap T_0)$ is a measure. It is an integral measure. We define map f such that $f(P) = 1$ for $P \in T - T_0$, $f(P) = 0$ for P on the surface of T, and $f(P) = -1$ for $P \in T_0 - T$. Then $\mu(R) = \int\int\int_R f dV$.