

Problem Set VI Solutions

1. §14.3 1ad.

(a) The mass M is given by $M = \int \int \int_R \delta dV = \int \int \int_R (x^2 + y^2 + z^2) dv$.

(d) Let line be the line containing points $(0, 0, 0)$ and $(1, 1, 1)$. The parametric equations of this line are $x = t, y = t$, and $z = t$. Let P be a point of coordinates (x, y, z) and let (t', t', t') be its projection on L . Then $t' = \frac{x+y+z}{3}$, since $((x-t)\hat{i} + (y-t)\hat{j} + (z-t)\hat{k}) \cdot (1, 1, 1)$. Hence the square of the distance from P to L is $r_P^2 = (x-t')^2 + (y-t')^2 + (z-t')^2 = x^2 + y^2 + z^2 - \frac{(x+y+z)^2}{3}$. So the moment of inertia about the line L is

$$\begin{aligned} I_L &= \int \int \int_R \delta r_P^2 dV = \\ &= \int \int \int_R (x^2 + y^2 + z^2)^2 - \frac{(x^2 + y^2 + z^2)(x + y + z)^2}{3} dV. \end{aligned}$$

2. §14.3 2c.

The distance of a point $P(x, y)$ to the line $y = 3$ is $r_P = |y - 3|$. Hence the moment of inertia about this line is

$$I_L = \int \int_R \delta r_P^2 dA = \int \int_R y(y - 3)^2 e^{xy} dA.$$

3. §14.3 3.

Denote the coordinates of P_0 by (x_0, y_0, z_0) . The volume of the sphere is equal to $\frac{4}{3}\pi a^3$. The average distance for P_0 to the points of S is given by

$$\begin{aligned} d_{av} &= \frac{1}{\int \int \int_S dV} \int \int \int_S P P_0 dV = \\ &= \frac{3}{4\pi a^3} \int \int \int_S \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} dV. \end{aligned}$$

4. §14.5 2b.

The moment of inertia is given by

$$\begin{aligned} I_z &= \int \int_R \delta(x^2 + y^2) dA = \frac{M}{a^2} \int \int_R (x^2 + y^2) dA = \\ &= \frac{M}{a^2} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) dx dy. \end{aligned}$$

Let us first calculate the moment of inertia for horizontal strips, i.e. hold y constant. Then we get an integral of a function of x , and integrating that we get the moment of inertia.

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) dx = ay^2 + \frac{a^3}{12}.$$

Then

$$I_z = \frac{M}{a^2} \int_{-\frac{a}{2}}^{\frac{a}{2}} (ay^2 + \frac{a^3}{12}) dy = \frac{M}{a^2} \frac{a^4}{6} = \frac{Ma^2}{6}.$$

5. 14.5 3.

Since $dV = \frac{4}{3}\pi r^3 = 4\pi r^2 dr$, the formula for the average distance becomes

$$\begin{aligned} d_{av} &= \frac{3}{4\pi a^3} \int \int \int_S r dV = \frac{3}{4\pi a^3} \int_0^a 4\pi r^3 dr = \\ &= \frac{3}{a^3} \int_0^a r^3 dr = \frac{3}{4}a \end{aligned}$$

6. §14.7 2a.

Suppose we take a new reference point O' . Then $\vec{R}(P)$ become a new function $\vec{R}'(P) = \vec{R}(P) + \vec{O'O}$. Hence

$$\begin{aligned} \vec{O'C}' &= \frac{\int \int \int_D \vec{R}' dV}{V} = \frac{\int \int \int_D (\vec{R} + \vec{O'O}) dV}{V} = \\ &= \frac{\int \int \int_D \vec{R} dV + (\vec{O'O})V}{V} = \vec{OC} + \vec{O'O} = \vec{O'C} \end{aligned}$$

Hence $\vec{O'C}' = \vec{O'C}$, so C and C' coincide, i.e. C doesn't depend on the choice of O .

7. §14.7 5.

If all the mass of M_1 were concentrated at P_1 and all the mass of M_2 were concentrated at P_2 , the center of mass of the M_1 and M_2 considered as a single mass would be a point P with

$$\vec{OP} = \frac{M_1\vec{OP}_1 + M_2\vec{OP}_2}{M_1 + M_2}.$$

Now let us see where the center of mass is in general. Let R be a region that contains both R_1 and R_2 . Let δ_1 and δ_2 be the density functions for M_1 and M_2 . Let f_1 be a function on R such that $f_1(Q) = \delta_1(Q)\vec{OQ}$ if $Q \in R_1$ and $f_1(Q) = 0$ otherwise. Let f_2 be a function on R such that $f_2(Q) = \delta_2(Q)\vec{OQ}$ if $Q \in R_2$ and $f_2(Q) = 0$ otherwise. Let f be a function on R such that $f(Q) = \delta_1(Q)\vec{OQ}$ if $Q \in R_1$, $f(Q) = \delta_2(Q)\vec{OQ}$ if $Q \in R_2$ and $f_2(Q) = 0$ otherwise. Since R_1 and R_2 are non-overlapping, $f(Q) = f_1(Q) + f_2(Q)$ for all points $Q \in R$. Then we have that

$$\vec{OP}_1 = \frac{1}{M_1} \int \int \int_{R_1} f_1(Q) dV \quad \text{and} \quad \vec{OP}_2 = \frac{1}{M_2} \int \int \int_{R_2} f_2(Q) dV$$

The center of mass P' in general is given by

$$\begin{aligned} \vec{OP}' &= \frac{1}{M_1 + M_2} \int \int \int_R f(Q) dV = \\ &= \frac{1}{M_1 + M_2} \left(\int \int \int_{R_1} f_1(Q) dV + \int \int \int_{R_2} f_2(Q) dV \right) = \\ &= \frac{M_1\vec{OP}_1 + M_2\vec{OP}_2}{M_1 + M_2}. \end{aligned}$$

Hence $\vec{OP} = \vec{OP}'$, so the centers of mass coincide.

8. §15.1 3a.

$$\begin{aligned} &\int_0^1 \int_x^{2x} \int_0^{xy} (x^2yz) dz dy dx = \\ &= \int_0^1 \int_x^{2x} \frac{x^4 y^3}{2} dy dx = \int_0^1 \frac{15x^8}{8} dx = \frac{15}{72} = \frac{5}{24}. \end{aligned}$$

9. §15.1 4a.

The region of integration is the region in \mathbf{E}^3 bounded by the planes of equations $x = 1$, $y = x$, and $y = 2x$ and the conical $z = xy$. Its base in the xy plane is the triangle of vertices $(0, 0)$, $(1, 1)$, and $(1, 2)$.

10. §15.2 2.

The region occupied by all points (x, y) when y goes from 0 to 2 and x goes from y to 2 is the triangle with vertices $(0, 0)$, $(0, 2)$, and $(2, 2)$. Hence we can say that x goes from 0 to 2 and y goes from 0 to x . By Fubini's Theorem, we have that

$$\int_0^2 \int_y^2 e^{x^2} dx dy = \int_0^2 \int_0^x e^{x^2} dy dx = \int_0^2 x e^{x^2} dx = \frac{1}{2}(e^4 - 1).$$

11. §15.4 2.

(a) Clearly x goes from 0 to 1, and y goes from 0 to $\sqrt{1-x^2}$. Since $x^2 + y^2 \leq 1$ we have that $x + y \leq 2$, so z goes from 0 to $1 - \frac{x+y}{2}$. The integral will be:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-\frac{x+y}{2}} dz dy dx$$

(b)

$$\begin{aligned} & \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{1-\frac{x+y}{2}} dz dy dx = \\ & = \int_0^1 \int_0^{\sqrt{1-x^2}} \left(1 - \frac{x+y}{2}\right) dy dx = \\ & = \int_0^1 \left(\left(1 - \frac{x}{2}\right)\sqrt{1-x^2} - \frac{1-x^2}{4}\right) dx = \\ & = -\frac{1}{6} + \int_0^1 \sqrt{1-x^2} dx + \int_0^1 -\frac{x}{2}\sqrt{1-x^2} dx = \\ & = \int_0^1 \sqrt{1-x^2} dx - \frac{1}{3} = \frac{\pi}{4} - \frac{1}{3} \end{aligned}$$

12. §15.6 5c.

Let the side of the base have length a and let the pyramid have altitude b . The intersection of a plane parallel to the base with the pyramid is clearly a square. If the plane is at distance z from the base plane, the side of this square will have length $\frac{b-z}{b}a$. We will calculate the moment of

inertia as an integral of the moment of inertia given by this square, using that the volume of a pyramid is $V = \frac{a^2 b}{3}$.

$$\begin{aligned} I_z &= \frac{M}{V} \int_0^b \int_{\frac{z-b}{2b}a}^{\frac{b-z}{2b}a} \int_{\frac{z-b}{2b}a}^{\frac{b-z}{2b}a} (x^2 + y^2) dx dy dz = \\ &= \frac{M}{V} \int_0^b \frac{(b-z)^4 a^4}{6b^4} dz = \frac{M a^4}{V b^4} \int_0^b \frac{z^4}{6} dz = \frac{M a^4 b}{V 30} = \frac{M a^2}{10}. \end{aligned}$$