Lecture IV
Analytic Geometry in $\mathbf{E^2}$ and $\mathbf{E^3}$

First we review some basic facts of analytic geometry in $\mathbf{E^2}$. Let us consider a Cartesian coordinate system in $\mathbf{E^2}$. We denote by $\mathscr{F}[x, y]$ an algebraic formula in the variables x and y. Any equation of the form $\mathscr{F}_1[x, y] = \mathscr{F}_2[x, y]$ can be reduced to an equation of the form $\mathscr{F}[x, y] = 0$. We say that an equation $\mathscr{F}[x, y] = 0$ determines a set S if S is the set of all points in $\mathbf{E^2}$ whose coordinates satisfy the equation.

In $\mathbf{E^2}$, an equation is called linear if has the form $Ax + By + C = 0$. where A, B, and C are real coefficients, with $A^2 + B^2 > 0$. Also, the following facts are true:

(a) Every linear equation determines a unique straight line.

(b) Every straight line is determined by some linear equation.

Let $\vec{N} = A\hat{\imath} + B\hat{\jmath}$. Then $\vec{N}$ is normal to the line L determined by $Ax + By + C = 0$. The vector $\hat{N} = \frac{\vec{N}}{|\vec{N}|}$ is a unit vector normal to the same line. Let O be the origin of the Cartesian system, let P be a point with position vector $\vec{R}_P$. Denote by d the distance from O to the line L. The following lemma holds:

Lemma 1 *If P is on the line L then $|\vec{R_P} \cdot \hat{N}| = d$.*

This result follows quickly from the equality $\vec{R_P} \cdot \hat{N} = |\vec{R_P}| \cos\theta$, where θ is the angle between $\vec{R_P}$ and $\hat{N}$.

We can use lemma 1 to find the distance from O to the line L. Let $R = (x, y)$ be a point on L and let $\vec{R} = x\hat{\imath} + y\hat{\jmath}$. Then $|C| = |Ax + By| = |\vec{R} \cdot \vec{N}| = |\vec{N}|d = \sqrt{A^2 + B^2}d$. Hence we get the following result:

Theorem 1 *The distance d between the origin O and the line determined by $Ax + By + C = 0$ is given by $d = \frac{|C|}{\sqrt{A^2+B^2}}$.*

In $\mathbf{E^2}$, an equation $\mathscr{F}[x, y] =$ is called a *second-degree equation* if $\mathscr{F}[x, y] = Ax^2 + By^2 + Cxy + Dx + Ey + F$, where A, B, C, D, E, and F are real and $A^2 + B^2 + C^2 > 0$.

As we go from $\mathbf{E^2}$ to $\mathbf{E^3}$, here is how the basic facts change:

An equation is called linear if it has the form $Ax + By + Cz + D = 0$, where A, B, C, and D are real coefficients, with $A^2 + B^2 + C^2 > 0$. In $\mathbf{E^3}$, a linear equation determines a unique plane, not a line.

The vectors $\vec{N} = A\hat{\imath} + B\hat{\jmath} + C\hat{k}$ and $\hat{N} = \frac{\vec{N}}{|\vec{N}|}$ are normal to the plane M determined by $Ax + \mathrm{B}y + Cz + D = 0$. Let d be the distance between the origin O and the plane M. Consider a point P with position vector $\vec{R}_P$. The following properties hold:

Lemma 2 *If P is in the plane M then $|\vec{R_P} \cdot \hat{N}| = d$.*

Theorem 2 *The distance d between the origin O and the plane M is given by:* $d = \frac{|D|}{\sqrt{A^2+B^2+C^2}}$.

Let M and M' be two planes determined by the equations $Ax+By+Cz+D = 0$ and $A'x + B'y + C'z + D' = 0$. Then:

(a) If M and M' are parallel, then there exists a real c such that $A = cA'$, $B = cB'$, and $C = cC'$.

(b) If M and M' aren't parallel, the there exists a point P with coordinates (x, y, z) such that $Ax + By + Cz + D = 0$ and $A'x + B'y + C'z + D' = 0$.

In $\mathbf{E^3}$, a system of two linear equations for two non-parallel planes determines a unique line. But more frequently parametric equations are used to determine lines in $\mathbf{E^3}$.

The system of equations

$$x = a_1 + tb_1, y = a_2 + tb_2, z = a_3 + tb_3$$

with t in $(-\infty, +\infty)$ is called a *system of linear scalar parametric equations on* $(-\infty, +\infty)$ in $\mathbf{E^3}$.

We say that a figure S is determined by a system of parametric equations on $(-\infty, +\infty)$ if S is the set of all points in $\mathbf{E^3}$ whose coordinates satisfy the equations as the *parameter* t takes all real values. Then each system of linear scalar parametric equations in $\mathbf{E^3}$ determines a unique line.

The vector interperpretation of this system of equations gives a clearer reasoning for the property above. Let $\vec{R_0}$ be the position vector of the point P_0 with coordinates (a_1, a_2, a_3), and let $\vec{A} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$. Then the set S of all points R with position vector $\vec{R} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ such that $\vec{R} = \vec{R_0} + t\vec{A}$ for real t is a line in $\mathbf{E^3}$.