

Lecture XXVII

Stokes's Theorem

In the previous lecture, we saw how Green's theorem deals with integrals on elementary regions in \mathbf{E}^2 and their boundaries. We also saw how the divergence theorem deals with elementary regions in \mathbf{E}^3 and their boundaries. We will now look at Stokes's theorem, which applies to integrals on elementary surfaces in \mathbf{E}^3 and their boundaries.

First let us extend the concept of rotational derivatives to \mathbf{E}^3 .

Definition 1 Let \vec{F} be a vector field with a domain D in \mathbf{E}^3 , and let P be an interior point of D . Given a unit vector \hat{u} in \mathbf{E}^3 let M be the plane that goes through P and is normal to \hat{u} . Consider this plane directed, with the direction given by \hat{u} . For a > 0 , let C_a be the circle in M of radius a and center P . Consider the limit

$$\lim_{a \rightarrow 0} \frac{1}{\pi a^2} \oint_{C_a} \vec{F} \cdot d\vec{R}$$

If the limit exists, then it is called the rotational derivative of \vec{F} in the direction \hat{u} at P and it is denoted by $\text{rot}\vec{F}|_{\hat{u},P}$.

If \vec{F} and \hat{u} are given in Cartesian coordinates, then the value of $\text{rot}\vec{F}|_{\hat{u},P}$ can be easily found out from the given formulas. Let $\vec{F} = L(x, y, z)\hat{i} + M(x, y, z)\hat{j} + N(x, y, z)\hat{k}$ and let $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$. Then

$$\text{rot}\vec{F}|_{\hat{u},P} = (N_j - M_z)u_1 + (L_z - N_x)u_2 + (M_x - L_y)u_3.$$

Definition 2 Let $\vec{F} = L(x, y, z)\hat{i} + M(x, y, z)\hat{j} + N(x, y, z)\hat{k}$ and let P be a point in the domain of \vec{F} . The vector \vec{C}_P , called the curl of \vec{F} at P and denoted by $\vec{\text{curl}}\vec{F}|_P$, is defined by the formula

$$\vec{\text{curl}}\vec{F}|_P = (N_y - M_z)\hat{i} + (L_z - N_x)\hat{j} + (M_x - L_y)\hat{k}$$

The curl of \vec{F} can also be represented by a determinant:

$$\vec{\text{curl}}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ L & M & N \end{vmatrix}$$

The following equality results immediately from the formulas given above.

Theorem 1 (Curl theorem)

$$\text{rot}\vec{F}|_{\hat{u},P} = \vec{\text{curl}}\vec{F}|_P \cdot \hat{u}$$

From the equality, since \hat{u} is a unit vector, it follows that the magnitude of $\vec{\text{curl}}\vec{F}|_P$ is the maximum value of $\text{rot}\vec{F}|_{\hat{u},P}$ as \hat{u} varies at P . The maximum is obtained when the direction of \hat{u} coincides with the direction of $\vec{\text{curl}}\vec{F}|_P$.

Theorem 2 (Stokes's theorem) *Let S be a two-sided directed surface whose boundary consists of the simple, closed curves C_1, \dots, C_k . Let \vec{F} be a vector field whose domain contains S . If we take the direction of each curve C_i to be coherent with the direction of S , then*

$$\int \int_S \vec{\text{curl}}\vec{F} \cdot d\vec{\sigma} = \oint_{C_1} \vec{F} \cdot d\vec{R} + \dots + \oint_{C_k} \vec{F} \cdot d\vec{R}$$