Lecture XVII

Curvilinear Coordinates; Change of Variables

As we saw in lecture 16, in $\mathbf{E^2}$ we can use the polar coordinates system. In this system, we have a fixed point O and a fixed ray $\vec{Ox}$. The coordinates of a point P are given by r, the distance from P to O, and θ the angle made by $\vec{OP}$ and $\vec{Ox}$, as measured going counterclockwise from $\vec{Ox}$ to $\vec{OP}$. We can change the system of coordiantes from polar to Cartesian through a system of equations:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

This is called the *defining system*. To go from the Cartesian system to the polar one, we use the *inverse defining system*:

$$r = \sqrt{x^2 + y^2}, \theta = \arctan\frac{y}{x},$$

for (x, y) in the first quadrant. Generally, we can introduce new non-Cartesian coordinates u, v by writing x, y as functions of these new coordiantes:

$$x = g(u, v), \qquad y = h(u, v).$$

These equations form the *defining system* for the new coordiantes. They can be summarized by writting the position vector

$$\vec{R}(u, v) = x\hat{i} + y\hat{j} = g(u, v)\hat{i} + h(u, v)\hat{j}.$$

In working with curvilinear coordinates, it is useful to introduce *unit coordinate vectors* $\hat{u}_P, \hat{v}_P$. To find $\hat{u}_P$, we take

$$\frac{\partial\vec{R}}{\partial u}|_P = \frac{\partial g}{\partial u}|_P\hat{i} + \frac{\partial h}{\partial u}|_P\hat{j},$$

and we form the unit vector $\hat{u}_P = \frac{\partial\vec{R}}{\partial u}|_P / |\frac{\partial\vec{R}}{\partial u}|_P|$. Similarly, we define $\hat{v}_P = \frac{\partial\vec{R}}{\partial v}|_P / |\frac{\partial\vec{R}}{\partial v}|_P|$. For example, recall that for polar coordinates, the defining system is $x = r\cos\theta, y = r\sin\theta$. The position vector is:

$$\vec{R} = r\cos\theta\hat{i} + r\cos\theta\hat{j},$$

and the unit coordinate vectors are:

$$\frac{\partial \vec{R}}{\partial r} = \cos\theta \hat{\imath} + \sin\theta \hat{\jmath} = \hat{r},$$

$$\frac{\partial \vec{R}}{\partial \theta} = -\sin\theta \hat{\imath} + \cos\theta \hat{\jmath} = \hat{\theta}.$$

In the uv plane, consider the rectangle determined by the vectors $\hat{u}\Delta u$ and $\hat{v}\Delta v$ with point P as their tail, where Δu and Δv are two small positive quantities. Now consider the point P' correspoding to P and the region corresponding to the rectangle in the xy plane. The area of this region is given by

$$\left| \frac{\partial \vec{R}}{\partial u}|_P \Delta u \times \frac{\partial \vec{R}}{\partial v}|_P \Delta v \right| = \left| \frac{\partial \vec{R}}{\partial u}|_P \times \frac{\partial \vec{R}}{\partial v}|_P \right| \Delta u \Delta v$$

This vector product can be easily computed, since

$$\frac{\partial \vec{R}}{\partial u} = \frac{\partial g}{\partial u}\hat{\imath} + \frac{\partial h}{\partial u}\hat{\jmath} \qquad and \frac{\partial \vec{R}}{\partial v} = \frac{\partial g}{\partial v}\hat{\imath} + \frac{\partial h}{\partial v}\hat{\jmath}.$$

Hence

$$\frac{\partial \vec{R}}{\partial u} \times \frac{\partial \vec{R}}{\partial v} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial(x,y)}{\partial(u,v)}.$$

This expression is called the *Jacobian of x and y with respect to u and v.* For polar coordinates, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\ cos^2\theta + r\sin^2\theta = r.$$

When switching from the Cartesian system to a curvilinear one, u, v might be given as functions of x, y, so the following equality is useful:

$$\frac{\partial(u,v)}{\partial(x,y)} = 1 \Big/ \frac{\partial(x,y)}{\partial(u,v)}\ .$$

The notion of Jacobian can be extended to $\mathbf{E^3}$ in the following manner. If x, y, z are functions of u, v, w, the Jacobian is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

For spherical coordinates ρ, φ, θ, the Jacobian is:

$$\frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} = r^2 \sin\varphi.$$

Jacobians are particularly useful when we compute integrals, because we can change variables in the following way:

$$\int\int_R f(x, y)dxdy = \int\int_{\hat{R}} f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv.$$