Lecture XXVIII

Measures; Irrotational fields

1 Circulation and flux measures

Let us first see what the theorems from the past lectures say about the circulation and flux measures. From Green's theorem, we get that, if $\vec{F}$ is a C^1 vector field on D in $\mathbf{E^2}$, then

$$\mu_{\vec{F}}(R) = \int \int_R rot\vec{F} dA$$

for every regular region R of D, where $\mu_{\vec{F}}$ is the circulation measure given by $\vec{F}$. In $\mathbf{E^3}$, from the divergence theorem we have that

$$\mu^f_{\vec{F}}(R) = \int \int \int_R div\vec{F} dV,$$

for every regular region R of D, where D is in $\mathbf{E^3}$ and $\mu^f_{\vec{F}}$ is the flux measure given by $\vec{F}$. From Stokes's theorem, we have

$$\mu^c_{\vec{F}}(S) = \int \int_S \vec{curl}\vec{F} \cdot d\vec{\sigma},$$

where $\mu^c_{\vec{F}}$ is the circulation measure given by $\vec{F}$. Remember that in $\mathbf{E^2}$, if $\vec{F} = L\hat{i} + M\hat{j}$, then

$$rot\vec{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

In $\mathbf{E^3}$, if $\vec{F} = L\hat{i} + M\hat{j} + N\hat{k}$, then

$$div\vec{F} = \frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}$$

$$\vec{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ L & M & N \end{vmatrix}$$

2 Irrotational fields

Definition 1 *A vector field* $\vec{F}$ *on* D *in* $\mathbf{E^2}$ *such that* $rot\vec{F} = 0$ *everywhere is called an* irrotational field on D.

If $\vec{F}$ is an irrotational field in $\mathbf{E^2}$ and C_1 and C_2 are two simple, closed, piecewise smooth curves having the same direction, such that C_2 is contained in C_1, then by Green's theorem we have

$$\oint_{C_1} \vec{F} \cdot d\vec{R} = \oint_{C_2} \vec{F} \cdot d\vec{R}.$$

Definition 2 *A vector field* $\vec{F}$ *on* D *in* $\mathbf{E^3}$ *such that* $div\vec{F} = 0$ *everywhere is called an* divergenceless field on D.

Let $\vec{F}$ be a divergenceless field in $\mathbf{E^3}$. Let S_1 and S_2 be two directed finite surfaces with identical boundaries, such that their directions are coherent with the directions of their boundaries. Then by the divergence theorem,

$$\int\int_{S_1} \vec{F} \cdot d\vec{\sigma} = \int\int_{S_2} \vec{F} \cdot d\vec{\sigma}.$$

Definition 3 *A vector field* $\vec{F}$ *on* D *in* $\mathbf{E^3}$ *such that* $\vec{curl}\vec{F} = 0$ *everywhere is called an* irrotational field on D.

Let $\vec{F}$ be an irrotational field in $\mathbf{E^3}$. Let S be a piecewise C^2 smooth, two-sided surface S with C and C' its boundary curves, both directed outward. Then by Stokes's theorem, we have

$$\oint_{C} \vec{F} \cdot d\vec{R} = \oint_{C'} \vec{F} \cdot d\vec{R}.$$

In $\mathbf{E^3}$, let us introduce the differential operator *del*, denoted by $\vec{\nabla}$ and defined by

$$\vec{\nabla} = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j} + \frac{d}{dz}\hat{k}.$$

If $\vec{F} = L\hat{i} + M\hat{j} + N\hat{k}$, then

$$\vec{\nabla} \cdot \vec{F} = \frac{dL}{dx} + \frac{dM}{dy} + \frac{dN}{dz} = div\vec{F}$$

By Stokes's theorem, if $\vec{F}$ is a C^2 vector field in $\mathbf{E^3}$, then

$$\vec{\nabla} \cdot \vec{curl}\vec{F} = 0.$$