18.024 Homework 1 - Solutions

Problem 1.

a) P. 457: 25.

If $A \perp B$, $A \cdot B = 0$. Then $\|A + xB\|^2 = A \cdot A + 2xA \cdot B + x^2 B \cdot B = A \cdot A + x^2 B \cdot B \geq A \cdot A = \|A\|^2$.

If $\|A + xB\| \geq A \cdot A = \|A\|$, we get that $2xA \cdot B + x^2 B \cdot B \geq 0$. Checking the cases $x > 0$, $x < 0$ and $x = 0$ gives that the only possibility for this to be true for **all** x is if $A \cdot B = 0$, ie. $A \perp B$.

b) P. 468: 15.

(a) Lets take a linear combination of $A+B$, $B+C$ and $C+A$ that adds to O. These vectors are linear independent if and only if they span O uniquely (with coefficients all equal to 0).

$$O = d(A+B) + e(B+C) + f(C+A) = (d+f)A + (d+e)B + (e+f)C$$

Since A, B and C are linearly independent, they span O uniquely, so $d+f = d+e = e+f = 0$. Solving this system we get that $d = e = f = 0$, so $A+B$, $B+C$ and $C+A$ span 0 uniquely and thus are linearly independent.

(b) Note that

$$1 \cdot (A - B) + 1 \cdot (B + C) + (-1) \cdot (C + A) = O$$

Hence, these vectors are linearly dependent.

Problem 2. P. 613: 4.

$$\begin{pmatrix} 3 & 2 & 1 & 1 \\ 5 & 3 & 3 & 2 \\ 7 & 4 & 5 & 3 \\ 1 & 1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, the solutions are of the form $X = (1, -1, 0) + t(-3, 4, 1)$.

Problem 3.

The idea is to use induction on the dimension of W to prove that W has an orthogonal basis. When the dimension is 1, any vector that belongs to W satisfies the property.

Now, assume that it is true if $dim(W) = k$. Consider W with dimension $(k+1)$. It has at least one basis: $\{A_1, A_2, \cdots, A_{k+1}\}$ with all the A_i linearly independent. Consider $W' = L(\{A_1, A_2, \cdots A_k\})$. It's dimension is k, so it has an orthogonal basis: $\{B_1, B_2, \cdots B_k\}$. Construct $B_{k+1} = A_{k+1} - \sum_{i=1}^{k} \frac{A_{k+1} \cdot B_i}{\|B_i\|^2}$.
It is easy to show that B_{k+1} is orthogonal to every B_i and that $\{B_1, B_2, \cdots B_{k+1}\}$ is an orthogonal basis of W. For the orthonormal basis, just take the unit vectors $C_i = \frac{B_i}{\|B_i\|}$.