18.034 Problem Set #8 due Th. 05/03/07 by noon,

Part I. Limit Cycles. Please consult class notes or the 04/27/07 handout, Chapter LC. Limit Cycles in 18.03 Lecture Notes.

Problem 1. We consider the plane autonomous system

$$x' = -y + x(1 - x^2 - y^2), \qquad y' = x + y(1 - x^2 - y^2). \tag{1}$$

(a) Show that $(0, 0)$ is the only critical point.

(b) Introduce the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Show that r and θ satisfy

$$r' = r(1 - r^2), \qquad \theta' = 1. \tag{2}$$

(c) Solve (2) with the initial data $r(0) = r_0, \theta(0) = \theta_0$. Use these, solve (1) with the initial data $x(0) = x_0$, $y(0) = y_0$.

(d) Discuss the trajectories for different values of r and argue that there is a periodic solution. What is the periodic solution?

Problem 2. pp. 167; #2(a).

Problem 3. Show that the plane autonomous system

$$x' = x + y^2 + x^3, \qquad y' = -x + y + yx^2$$

has no nontrivial periodic solution.

Part II. Fourier Series

Problem 4. The function f is defined as

$$f(x) = \begin{cases} 1 & \text{for } -2 \le x < 0, \\ x & \text{for } 0 \le x \le 2. \end{cases}$$

(a) Compute the Fourier series for f on the interval $-2 \le x \le 2$.

(b) Show that the series in part (a) converges to

$$\begin{cases} 1 & \text{if } -2 < x < 0, \\ x & \text{if } 0 < x < 2, \\ \frac{1}{2} & \text{if } x = 0, \\ \frac{3}{2} & \text{if } x = \pm 2. \end{cases}$$

(c) Use part (b) to show the identity

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Problem 5. Expand $f(x) = 1$ on $0 \le x \le \pi$ into a uniformly convergent sine series.