Pset #1 Solutions

1. We have that
$$\left(\frac{1}{y}\right)' = -\frac{y'}{y^2} = \frac{1-y^2}{y^2} = \left(\frac{1}{y}\right)^2 - 1.$$
Also, if $g(x) = -f(-x)$, then $g'(x) = (-1)\cdot(-f'(-x)) = f'(x)$, so as $g(x)^2 = f(x)^2$ we see
$$g'(s) = f'(-x) = f(-x)^2 - 1 = g(x)^2 - 1$$
as we wanted.

2. (a) As $y^2 + 1 > 0$ no matter what y is, $y' > 0$ and y is an increasing function. This is separable, so we rewrite the given equation as
$$\frac{dy}{y^2+1} = dx$$
and integrate to obtain $x = \arctan y + c$ for some constant c.

   (b) All solutions thus take the form of $y = \tan(x - c)$, and so are defined only on the interval $(c-\pi/2, c+\pi/2)$, an interval of length π.

3. (a) We can use an integrating factor of e^{-x} (which never vanishes), obtaining the equation
$$xe^{-x}y' + \left(e^{-x} - xe^{-x}\right)y = (xe^{-x}y)' = 0.$$
We thus obtain integral curves from $xe^{-x}y = C$, giving the solution $y = Ce^x/x$ (note that this is not defined for $x = 0$ unless $C = 0$).

   (b) If $y(1) = 1$, then $Ce^1/1 = 1$ and $C = 1/e$, so $y = e^{x-1}/x$. If $y(1) = 2$, then $Ce^1/1 = 2$, $C = 2/e$, and $y = 2e^{x-1}/x$.

4. (a) We expect this to have a quadratic solution $y = Ax^2 + Bx + C$, in which case $y' = 2Ax + B$ and
$$x^2+4x+7 = 2Ax+B+2Ax^2+2Bx+2C = 2Ax^2+2(A+B)x+(B+2C).$$
But then $A = \frac{1}{2}$, $B = 2 - A = \frac{3}{2}$, and $C = (7-B)/2 = \frac{11}{4}$. Our polynomial solution is then
$$y = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{11}{4}.$$

(b) The homogeneous equation $y_h' + 2y_h = 0$ has general solution $y_h = Ce^{-2x}$, so we need to solve for C in $y = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{11}{4} + Ce^x$ subject to $y(0) = 0$. This just means $\frac{11}{4} + C = 0$, so the solution we seek is

$$y = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{11}{4} - \frac{11}{4}e^x.$$

5. (a) Setting aside the solution $y = 0$ (or assuming $n = 0$), we divide both sides by y^n to obtain

$$\frac{y'}{y^n} + p(x)y^{1-n} = q(x).$$

If $u = y^{1-n}$, then $u' = (1-n)y^{-n}$, and so (as $n \neq 1$) we get

$$\frac{u'}{1-n} + p(x)u = q(x),$$

a linear ODE as we wanted.

(b) Rewriting this as $u' + (1-n)p(x)u = (1-n)q(x)$, we use the usual method, an integrating factor of $e^{\int(1-n)p(x)dx}$, so that we get

$$\left(e^{\int(1-n)p(x)dx}u\right)' = (1-n)q(x)e^{\int(1-n)p(x)dx},$$

so that we can solve for u:

$$u = e^{-\int(1-n)p(x)dx}\left(\int(1-n)q(x)e^{\int(1-n)p(x)dx}dx + C\right).$$

Finally, use $y = u^{1/(1-n)}$ to get

$$y = e^{-\int p(x)dx}\left(\int(1-n)q(x)e^{\int(1-n)p(x)dx}dx + C\right)^{\frac{1}{1-n}}.$$

6. If u and v are both integrals, then they are both constant on all solution curves of the DE, whence $u+v$, uv, $\lambda u + \mu v$ and $g(u)$ will all be constant on all solution curves as well. More precisely, for any two points (x_1, y_1) and $x_2, y_2)$ on the same solution curve, $u(x_1, y_1) = u(x_2, y_2)$ and $v(x_1, y_1) = v(x_2, y_2)$, so for example

$$uv(x_1, y_1) = u(x_1, y_1)v(x_1, y_1) = u(x_2, y_2)v(x_2, y_2) = uv(x_2, y_2).$$

7. (a) Using the substitution $v = y/x$, $y' = (xv)' = xv' + v$, so $xv' = \sin v - v$ and

$$\frac{v'}{\sin v - v} = \frac{1}{x}.$$

Integrating, we obtain

$$x = Ce^{\int \frac{dv}{\sin v - v}},$$

which at least characterizes all solutions implicitly as x in terms of y/x (the integral cannot be computed explicitly).

Alternatively, we can use polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, so that

$$y' = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}.$$

Rearranging this expression a bit, we see that

$$\sin\tan\theta = y' = \frac{\frac{1}{r}\frac{dr}{d\theta}\tan\theta + 1}{\frac{1}{r}\frac{dr}{d\theta} - \tan\theta}.$$

We may then solve for $\frac{1}{r}\frac{dr}{d\theta}$ to obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{(\sin\tan\theta)\tan\theta + 1}{\sin\tan\theta - \tan\theta}.$$

Multiplying both sides by $d\theta$ and integrating, we obtain the general solution of

$$r(\theta) = Ce^{\int \frac{(\sin\tan\theta)\tan\theta + 1}{\sin\tan\theta - \tan\theta} d\theta}.$$

(b) Suppose $y = f(x)$ is a solution. Then if we set $ky = f(kx)$ (scaling by a factor of the nonzero constant k), we are investigating whether $y = f(kx)/k$ is also a solution. But here

$$y' = k \cdot \frac{f'(kx)}{k} = f'(kx) = g(\frac{f(kx)}{kx}) = g(\frac{f(kx)/k}{x}) = g(\frac{y}{x})$$

as we wanted.