18.034 Problem Set #5 due Th. 03/22/07 by noon

Problem 1. Laplace transform of t^r. (a) Use the well-known formula

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

to show that $\mathcal{L}[t^{-1/2}] = \sqrt{\pi/s}$, $s > 0$.

(b) Deduce from the part (a) that $\mathcal{L}[t^{1/2}] = (\sqrt{\pi}/2)s^{3/2}$, $s > 0$.

In order to compute $\mathcal{L}[t^r]$ for arbitrary r, we make use of the *Gamma function*, defined by the integral

$$\Gamma(r+1) = \int_0^\infty e^{-t}t^r dt.$$

(c) Show that the improper integral converges for all r.

(d) Show that $\Gamma(r+1) = r\Gamma(r)$ for $r > 0$. Show that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$.

(e) For $r > -1$, show that $\mathcal{L}[t^r] = \Gamma(r+1)/s^{r+1}$, $s > 0$.

Problem 2. Verify that $\mathcal{L}[(\sin t)/t] = \arctan(1/s)$, $s > 1$. (Hint: Use the Taylor series for $\sin t$.)

Problem 3. Consider the Bessel equation of order zero

(*) $$ty'' + y' + ty = 0.$$

Note that $t = 0$ is a singular point, and thus solutions may become unbounded as $t \to 0$. Nevertheless, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assume that there is such a solution $y = y(t)$ and denote $Y(s) = \mathcal{L}[y](s)$.

(a) Show that $Y(s)$ satisfies

$$(1+s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1+s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Expand $(1+s^2)^{-1/2}$ in a binomial series for $s > 1$. Assuming that it is permissible to take the inverse transform term by term, show that

$$y(t) = c\sum_{n=0}^\infty \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2},$$

which is referred to as the *Bessel function* of the first kind of order zero. Show that $y(0) = 1$ and y has finite derivatives of all orders at $t = 0$.

Problem 4. Resonance and Beats. An undamped harmonic oscillator such as a spring-mass system with a sinusoidal forcing term experiences resonance if the frequency of the forcing term is the same as the natural frequency. If the forcing frequency is slightly different from the natural frequency, then the system exhibits a beat. In this problem, we explore the effect of some nonsinusoidal forcing functions.

(a) Solve the initial value problem

$$y'' + y = u_0(t) + 2\sum_{k=1}^n (-1)^k u_{k\pi}(t), \qquad y(0) = 0, \quad y'(0) = 0,$$

where $u_c(t)$ is the unit step function with the jump discontinuity by the unity at $t = c$.

(b) Investigate how the solution changes as $n \to \infty$.

(c) Solve the initial value problem

$$y'' + y = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}(t), \qquad y(0) = 0, \quad y'(0) = 0.$$

(d) Compare the results of part (a) and (c) in the link with *resonance* and *beats*.

Problem 5. Volterra integral equation. Consider the integral equation

$$\phi(t) + \int_0^t (t-\tau)\phi(\tau)d\tau = \sin 2t.$$

(a) Show that the given integral equation is equivalent to the initial value problem

$$u'' + u = \sin 2t, \qquad u(0) = 0, \quad u'(0) = 0.$$

(b) Solve the integral equation by using the Laplace transform.

(c) Solve the initial value problem of part (a) and verify that the solution is the same as that obtained in part (b).