

## Pset # 7 Solutions

1. The Jacobian here is

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & -1 \end{pmatrix},$$

which has eigenvalues  $\pm\sqrt{\frac{3}{2}}$ . Thus this is an unstable critical point, a saddle.

2. If we use the Lyapunov function  $E(x, y) = x^2 + y^2$ , we obtain

$$\dot{E} = 2x^6 + 2xy - 2xy + 2y^4 = 2x^6 + 2y^4,$$

so that all solutions have strictly increasing magnitude as  $t$  increases; this demonstrates that the system is not stable (or strictly stable).

3. If  $\dot{x} = \sin y = 0$ , then we must have  $y = k\pi$  for some integer  $k$ . This means that  $\dot{y} = \sin x + \sin y = \sin x = 0$ , so that  $x = \ell\pi$  for some other integer  $\ell$ . At the critical point  $(\ell\pi, k\pi)$  we thus have a Jacobian of

$$\begin{pmatrix} 0 & (-1)^k \\ (-1)^\ell & (-1)^k \end{pmatrix}$$

, which has eigenvalues  $(-1)^k$  and  $(-1)^\ell$ . We conclude that this critical point is an unstable (source) node if  $k$  and  $\ell$  are even, a stable (sink) node if  $k$  and  $\ell$  are odd, and an unstable (saddle) critical point if  $k$  and  $\ell$  are of opposite parity.

4. Here  $F'(0)$  is the Jacobian of the critical point  $x = 0$ , and that  $F'(0) < 0$  simply says that the eigenvalue of this one-term matrix is negative, implying that the critical point is asymptotically stable as usual.
5. We can use the Lyapunov function  $E(x, y) = x^4 + 2y^2$ , so that

$$\dot{E} = 4x^3(y - x^3) + 4y(-x^3) = -4x^6.$$

This is negative semi-definite, and so demonstrates that the critical point is stable. The Jacobian, however, is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

the solutions to the corresponding linearization being  $(x, y) = (a+bt, b)$ , which are unbounded in magnitude as  $t \rightarrow \infty$ .