Pset # 5 Solutions

1. (a) We use the substitution $u = \sqrt{st}$ (for $s > 0$ say), so that $du = \frac{1}{2}t^{-1/2}\sqrt{s}dt$, and compute

$$\int_0^\infty t^{-1/2}e^{-st}dt = \int_0^\infty 2s^{-1/2}e^{-u^2}du = \sqrt{\frac{\pi}{s}}.$$

   (b) As $t^{1/2}$ is twice the antiderivative of $t^{-1/2}$, the Laplace tranform of $t^{1/2}$ is $2/s$ times the Laplace transform of $t^{-1/2}$, namely $2\sqrt{\pi}s^{-3/2}$, as we wanted.

   (c) This converges for $r > -1$. To see this, we note that the integral from 0 to 1 (which has a positive integrand) is bounded above by t^r, and so converges as $\int_0^1 t^r dt = \frac{1}{r+1}$ is bounded. For the remainder, we use the integral test, comparing the integral to the series $\sum_{n=1}^\infty n^r e^{-n}$, and then using the ratio test here. The ratio $\frac{(n+1)^r e^{-n-1}}{n^r e^{-r}}$ converges to $1/e$ as $n \to \infty$, so that the series and hence the integral converges.

   (d) For the first part, we use integration by parts.

$$\Gamma(r+1) = \int_0^\infty t^r e^{-t}dt = rt^{r-1}e^{-t}\Big|_0^\infty + r\int_0^\infty t^{r-1}e^{-t}dt = r\Gamma(r).$$

   Moreover, $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$ and with the substitution $u = t^{1/2}$,

$$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}dt = \int_0^\infty 2e^{-u^2}du = \sqrt{\pi}.$$

   (e) We wish to compute $\int_0^\infty t^r e^{-st}dt$. Using the substitution $u = st$ for $s > 0$, we obtain

$$\int_0^\infty (u/s)^r e^- u\frac{1}{s}du = s^{-r-1}\int_0^\infty u^r e^{-u}du = \frac{\Gamma(r+1)}{s^{r+1}}.$$

2. We verify this term-by-term in the Taylor expansion for $\sin t/t$, namely

$$1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots .$$

Indeed, according to the previous problem, the Laplace transform of t^{2n} is $\Gamma(2n+1)/s^{2n+1}$. Now, again according to the previous problem,

$$\Gamma(2n+1) = 2n\Gamma(2n) = 2n \cdot (2n-1) \cdot (2n-2) \cdots 1 \cdot \Gamma(1) = (2n)!.$$

Therefore, the Laplace transform of $t^{2n}/(2n+1)!$ is $1/(2n+1)s^{2n+1}$. As the Taylor expansion of $\arctan u$ is

$$u - \frac{u^3}{3} + \frac{u^5}{5} - \cdots,$$

this shows that $\mathcal{L}[\sin t/t]$ is in fact $\arctan(1/s)$.

3. (a) Notice that if $x(t)$ is a function with Laplace transform $X(s)$, then

$$\mathcal{L}[tx](s) = \int_0^\infty tx(t)e^{-st}dt = -\frac{d}{ds}\int_0^\infty x(t)e^{-st}dt = -X'(s).$$

Therefore as we know $\mathcal{L}[y'] = sY$ and $[y''] = s^2Y$,

$$\begin{aligned} \prime &= L[ty'' + y' + ty] \\ &= -\frac{d}{ds}(s^2Y) + sY - Y' \\ &= -2sY - s^2Y' + sY - Y' \\ &= -[(s^2+1)Y' + sY]. \end{aligned}$$

(b) We separate variables to get

$$\frac{Y'}{Y} = -\frac{s}{s^2+1},$$

whence $\log Y = -\frac{1}{2}\log(1+s^2) + C$ and $Y = c(1+s^2)^{-1/2}$ as we wanted.

(c) To expand this in a binomial series, we compute

$$\begin{aligned} \binom{-\frac{1}{2}}{n} &= \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!} \\ &= \frac{(-1)^n \cdot 1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^n n!} \\ &= \frac{(-1)^n(2n)!}{2^n n! \cdot 2 \cdot 4 \cdot \cdots \cdot (2n)} \\ &= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}. \qquad (1) \end{aligned}$$

Now, we represent $c(1+s^2)^{-1/2}$ as $cs^{-1}(1+1/s^2)^{-1/2}$, obtaining

$$\mathcal{L}[y] = \frac{c}{s}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}s^{-2n} = c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\cdot\frac{(2n)!}{s^{2n+1}}.$$

As $(2n)! = \Gamma(2n+1)$, $(2n)!/s^{2n+1} = \mathcal{L}[t^{2n}]$, and we conclude that

$$y(t) = c\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$$

as desired. This series evidently converges faster than e^t and so is convergent on all of $\mathbb{R}$; we conclude that (setting $c = 1$), $y(0) = (-1)^0/(2^0\cdot(0!)^2) = 1$ and that the n^{th} derivative of y, which is just 0 or $n!$ times the $(n/2)^{\text{th}}$ term in the series above (if n is odd or even respectively), is finite.

4. (a) First we solve the problem $y'' + y = u_c(t)$ with $y(0) = y'(0) = 0$ and then use superposition to get us the rest of the way there. Indeed, for $t < c$ we must have $y(t) = 0$ by uniqueness, so by continuity we have $y(c) = y'(c) = 0$. But then to the right of c, $y = A\cos(t-\phi)+1$, so $y(c) = A\cos(c-\phi)+1 = 0$ and $y'(c) = -A\sin(c-\phi) = 0$. This yields (for example) $\phi = c$ and $A = -1$, so $y(t) = u_c(t)[1-\cos(t-c)]$. But then the solution we seek in the end is

$$y(t) = u_0(t)\cos t + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}[1-\cos(t-k\pi)].$$

(b) It is the case that $\cos(t-k\pi) = (-1)^k\cos t$, so we can actually write this solution as

$$y(t) = 2\sum_{k=1}^{\infty}u_{k\pi} + [1-2\sum_{k=1}^{\infty}u_{k\pi}]\cos t = 2[t/\pi] + (1-2[t/\pi])\cos t),$$

where in the last expression the brackets refer to the greatest integer function. This is a function which remains positive but which returns to the value $y = 1$ infinitely man times (so its maximum grows, but it always drops back down).

(c) As before, we get

$$y(t) = u_0(t)\cos t + 2\sum_{k=1}^{n}(-1)^k u_{11k/4}[1 - \cos(t - 11k/4)].$$

(d) If we look at $\cos(t - 11k/4)$ as the real part of $e^{(t-11k/4)i}$, in the interval $[11n/4, 11(n+1)/4)$ we are looking at the real part of

$$2n+1-2\sum_{k=1}^{n}(-1)^k e^{(t-11k/4)i} = 2n+1+2e^{it}\frac{1+(-1)^{n+1}e^{-11(n+1)i/4}}{1+e^{-11i/4}}.$$

On the right side of this expression, the denominator is a nonzero constant and the rest of the expression is bounded by 2, so the right side can never grow past a certain size. We conclude that the solution will eventually consistently exceed any number (unlike the first solution, above, which always returns to $y = 1$).

5. (a) If we set

$$u = \int_0^t (t-\tau)\phi(tau)d\tau = t\int_0^t \phi(\tau)d\tau - \int_0^t \tau\phi(\tau)d\tau,$$

then

$$u' = \int_0^t \phi(\tau)d\tau + t\phi(t) - t\phi(t) = \int_0^t \phi(\tau)d\tau)$$

so that $u'' = \phi(t)$. We conclude that we are solving

$$u'' + u = \sin(2t)$$

with the initial conditions that $u(0) = u'(0) = 0$ (as integrals from 0 to 0 are necessarily 0).

(b) Using the Laplace transform, we are looking to solve $(1+s^2)U(s) = 2/(s^2+4)$, so

$$U(s) = \frac{2}{(s^2+1)(s^2+4)} = \frac{2/3}{s^2+1} - \frac{2/3}{s^2+4},$$

so applying the inverse Laplace we obtain $u = \frac{2}{3}\sin t - \frac{1}{3}\sin(2t)$.

(c) A particular solution is given by $A\sin(2t)$, so that $-4A + A = 1$ and $A = -1/3$. The general solution is then $u = B\sin t + C\cos t - \frac{1}{3}\sin(2t)$, so that $u(0) = C = 0$ and $u'(0) = B - 2/3 = 0$. We conclude that $u(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin(2t)$, as we wanted.