Pset #2 Solutions

1. We need only show that $u(x, y) = 5x^2 + 6xy + 5y^2$ is an integral for the DE. But the derivative of u is just $(10x + 6y) + (6x + 10y)y'$, which is twice the given DE.

2. Using separation of variables, this reduces to the equation

$$\frac{dy}{y} = -\frac{dx}{x},$$

so integrating we obtain $\log y = -\log x + C$, or $y = C/x$ (for a different, still arbitrary, C). This comprises all solution curves, although for each $C \neq 0$ we actually have two solution curves, one for $x > 0$, the other for $x < 0$ (due to the function not being defined in this case at $x = 0$). If we instead solve this DE by means of the integral $u(x, y) = xy$ (whose derivative is just the DE in question), the integral curves $xy = C$ consist of the functions $y = C/x$ (defined for $x \neq 0$) whenever $C \neq 0$. If $C = 0$, we get more than the function $y = 0$, we get the y-axis as well. Therefore the integral curves are very close, but not identical, to the solution functions obtained from separation of variables.

3. We use the integral $u(x, y) = |x|x + |y|y$, whose derivative is twice the given DE. We thus want to look at the level curves $x|x| + y|y| = C$. If $x \geq 0$ and $y \geq 0$, this gives circular arcs $x^2 + y^2 = C$. If $x \geq 0$ and $y \leq 0$, then we get hyperbolic arcs $x^2 - y^2 = C$. If $x \leq 0$ and $y \geq 0$, we again get hyperbolic arcs $y^2 - x^2 = C$. Finally, if both x and y are nonpositive, then we get circular arcs $x^2 + y^2 = -C$. Putting this data together, we see that for $C > 0$ we have solution curves as follows:

$$y = \begin{cases} \sqrt{x^2 + C} & x < 0 \\ \sqrt{C - x^2} & 0 \leq x < \sqrt{C} \\ -\sqrt{x^2 - C} & \sqrt{C} \leq x, \end{cases}$$

for $C < 0$ we have solutions:

$$y = \begin{cases} \sqrt{x^2 + C} & x < \sqrt{-C} \\ -\sqrt{-C - x^2} & \sqrt{-C} \leq x < 0 \\ -\sqrt{x^2 - C} & 0 \leq x, \end{cases}$$

and for $C = 0$ we have the solution $y = -x$.

4. By ordinary differentiation,

$$y'' = (y')' = \frac{(c+dy')(ax+by)-(a+by')(cx+dy)}{(ax+by)^2} = (ad-bc)\frac{xy'-y}{(ax+by)^2}.$$

We may then substitute $y' = \frac{cx+dy}{ax+by}$ to obtain

$$y'' = (ad-bc)\frac{x(cx+dy)-y(ax+by)}{(ax+by)^3} = (ad-bc)\frac{cx^2+(d-a)xy-by^2}{(ax+by)^3}$$

as we wanted. As convexity of a solution is controlled by the sign of y'', it is enough to determine where $y'' > 0$ (so that y is convex) and where $y'' < 0$ (so that y is concave). The sign of y'' is the product of three signs, namely the sign of $(ad-bc)$, the sign of $ax+by$, and the sign of $cx^2+(d-a)xy-by^2$. This is perhaps easier to analyze in polar coordinates, so that as $r > 0$, we are looking at the signs of $(ad-bc)$, $a\cos\theta + b\sin\theta$, and $c\cos^2\theta + (d-a)\sin\theta\cos\theta - b\sin^2\theta$. The second term is $|a+bi|\cos(\theta - \arg(a+bi))$ and the third term is

$$\frac{c-b}{2} + \frac{c+b}{2}\cos 2\theta + \frac{d-a}{2}\sin 2\theta,$$

and so its sign is just the sign of

$$(c-b) + |c+b+(d-a)i|\cos(2\theta - \arg(c+b+(d-a)i)).$$

We can clean up this setup a bit by setting $\alpha = a+bi$ and $\beta = c+di$, so that $ad-bc = \mathrm{Im}(\alpha\beta)$,

$$|a+bi|\cos(\theta - \arg(a+bi)) = \mathrm{Re}\left(\alpha e^{-i\theta}\right),$$

and

$$\begin{aligned}(c-b)+|c+b &+ (d-a)i|\cos(2\theta - \arg(c+b+(d-a)i)) \\ &= \mathrm{Re}\left[(\beta+i\alpha)\left(1+e^{-2i\theta}\right)\right] \\ &= \mathrm{Re}\left[(\beta+i\alpha)\cos\theta e^{-i\theta}\right] \\ &= \cos\theta\mathrm{Re}\left[(\beta+i\alpha)\, e^{-i\theta}\right].\end{aligned}$$

Thus the critical angles, the rays where the sign of y'' changes sign, are those values of θ where $\alpha\beta$ is real (which doesn't happen by assumption), those points where θ is $\arg\alpha$ plus an odd multiple of $\pi/2$, where θ itself is an odd multiple of $\pi/2$, and those points where θ is $\arg(\beta + i\alpha)$ plus an odd multiple of $\pi/2$. In terms of slopes, the critical slopes are $-b/a$, ∞, and $\frac{b-c}{a+d}$.

5. (a) As $y_1' = 0 = \frac{3}{2}y_1^{1/3}$ and $y_2' = \frac{3}{2}x^{1/2} = \frac{3}{2}y_2^{1/3}$, this is immediate.

   (b) As the function $F(x, y) = \frac{3}{2}y^{1/3}$ satisfies the Lipschitz condition for $y > 0$, there can be only one solution curve with a positive initial y-value. More precisely, if $y(x_0) = y_0 > 0$ and y satisfies our DE, then $y(x) = (x - c)^{3/2}$ for x near x_0, where $c = x_0 - y_0^{2/3}$ is the unique value of c such that $y(x_0) = y_0$ (this is a solution, so it is the unique solution).

   I claim that if the nonnegative function y is strictly increasing for nonnegative x, then $y(x) = x^{3/2}$. Indeed, as $y(x) > 0$ for some $x > 0$, by the above remarks we know that there is some c such that $y(x) = (x - c)^{3/2}$. As in fact $y(x) > 0$ for *all* $x > 0$, we know that $(x - c)^{3/2} > 0$ for all $x > 0$, so $(0 - c)^{3/2} \geq 0$ and $c \leq 0$. Finally,

$$0 = y(0) = \lim_{x \to 0^+} y(x) = \lim_{x \to 0^+} (x - c)^{3/2} = (-c)^{3/2},$$

   so in fact $c = 0$ and the claim is proven.

   So now let $y(x)$ be a general solution, and let $c \geq 0$ be the supremum of all x such that $y(x) = 0$. If $c = +\infty$, then $y = y_1$. If $c < +\infty$, then for all $x > c$, $y'(x) = \frac{3}{2}y(x)^{1/3} > 0$, so y is strictly increasing for all $x \geq c$. But then $z(x) = y(x + c)$ is a solution to the same DE which is strictly increasing for nonnegative x and which has $z(0) = 0$, so $z(x) = x^{3/2}$ and $y(x) = z(x-c) = (x-c)^{3/2}$, completing the proof.

6. (a) We guess that there is a solution of the form $u = Ce^x$, and find that under this assumption

$$e^x = u'' + 3u' + 2u = 6Ce^x,$$

   so in fact $u_p = \frac{1}{6}e^x$ is a solution. We thus reduce (by superposition) to finding the general solution to the homogeneous equation $u_h'' + 3u_h' + 2u_h = 0$. We seek out exponential solutions first, trying $u_h = e^{\alpha x}$, so that $u_h' = \alpha e^{\alpha x}$, $u_h'' = \alpha^2 e^{\alpha x}$, and

$$0 = u_h'' + 3u_h' + 2u_h = (\alpha^2 + 3\alpha + 2)e^{\alpha x},$$

concluding that α can equal -1 or -2. This gives us the two independent solutions we wanted, so the general solution is

$$u = \frac{1}{6}e^x + c_1 e^{-x} + c_2 e^{-2x}.$$

(b) If $u(0) = 4$, then $\frac{1}{6} + c_1 + c_2 = 4$, and if $u'(0) = 4$, then $\frac{1}{6} - c_1 - 2c_2 = 4$. Adding these equations we get $\frac{1}{3} - c_2 = 8$, so $c_2 = -\frac{23}{3}$ and $c_1 = 4 - \frac{1}{6} + \frac{23}{3} = \frac{23}{2}$. Therefore the solution we seek is $u = \frac{1}{6} + \frac{23}{2}e^{-x} - \frac{23}{3}e^{-2x}$.