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Part I: TF Questions. Answer for each of the following statements if it is true or fale. Simply say **T** (if you believe it is true) or **F** (if you suspect it is false); you don't need to justify your answers. Each question counts 5pts.

1. Let the differential operator $L[y] = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$ where $a_1, \ldots, a_n$ are real numbers, have the characteristic polynomial $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots a_n$. If $p(\alpha) \neq 0$ then $L[y] = e^{\alpha x}$ has a solution of the form $Ae^{\alpha x}$.

2. The solution of the initial value problem $y''' + y'' - 2y = \delta(t)$ with $y(0) = y'(0) = y''(0) = 0$ decays exponentially as $t \to \infty$. Here, $\delta(t)$ denotes the Dirac delta function (or the unit impulse function).

3. The curves defined parametrically as solutions of the system

$$\frac{dx}{dt} = -\frac{y}{x^2}, \quad \frac{dy}{dt} = \frac{1}{x}$$

are orthogonal to the surface $y/x =$constant.

4. The function

$$f(x) = \begin{cases} x \log x & 0 < x \leq 1, \\ 0 & x = 0 \end{cases}$$

satisfies a Lipschitz condition.

5. The solution of the initial value problem $x' = t^2 + x^2$, $x(0) = 0$ exists for $|t| \leq 1/\sqrt{2}$.

6. All real-valued solutions of the DE $x'' + \sin x = b(t)$, where b is continuous $|t| < \infty$ exist for all real t.

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Part II: Proofs. Complete this part and submit your paper on **Thursday, April 19th 12–1pm in 2-230**. You have to deliver it yourself. Late submission is not allowed. You may consult the textbook, Supplementary notes, and your lecture notes. You are not allowed to discuss with other people, nor search other references.

Problem 1. Consider the equation

$$y^{(4)} - \lambda^4 y = 0,$$

where λ is a nonzero real constant. Observe that $\cos \lambda x$, $\sin \lambda x$, $\cosh \lambda x$, $\sinh \lambda x$ are solutions.

(a) (10pts) Show that there are nontrivial solutions ϕ satisfying

$$\begin{cases} \phi(0) = 0, & \phi'(0) = 0 \\ \phi(1) = 0, & \phi'(1) = 0 \end{cases} \tag{1}$$

if and only if $\cos \lambda \cosh \lambda = 1$.

(b) (10pts) Compute all nontrivial solutions satisfying (1).

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Problem 2. (10pts) Find the Laplace transform of $|\sin t|$ for $t > 0$.

Problem 3. Let f be a real-valued continuous function defined on the rectangle

$$R : |t - t_0| \leq T, \quad |x - x_0| \leq K$$

and let I be an interval of x containing x_0. Consider the initial value problem

(2) $$x'' = f(t, x), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

(a) (10pts) Show that ϕ is a solution of the initial value problem (2) on I if and only if ϕ is a solution on I of the integral equation

(3) $$x = x_0 + (t - t_0)x_1 + \int_{t_0}^{t} (t - \tau) f(\tau, x) d\tau.$$

(Hint. Use

$$\frac{d}{dt} \int_a^t F(\tau, t) d\tau = F(t, t) + \int_a^t \frac{\partial F}{\partial t}(\tau, t) d\tau.)$$

(continued)

(b) (10pts) Let $\phi_0, \phi_1, \phi_2, \ldots$ be successive approximations for (3). That is, $\phi_0(t) = x_0$ and

$$\phi_{n+1}(t) = x_0 + (t - t_0)x_1 + \int_{t_0}^{t} (t - \tau) f(\tau, \phi_n(\tau)) d\tau, \quad n = 0, 1, 2, \ldots.$$

Prove that the sequence ϕ_n satisfies

$$|\phi_n(t) - x_0| \leq M|t - t_0|$$

on the interval $|t - t_0| \leq \min(T, K/M)$, where $M = |x_1| + (T/2) \sup |f(t, x)|$ and deduce that $\{\phi_n\}$ converges on the interval $|t - t_0| \leq \min(T, K/M)$ to the initial value problem (2).

(c) (10pts) Suppose that f is a real-valued continuous function on the (t, x)-plane, $|t| < \infty$, $|x| < \infty$ and that f satisfies a Lipschitz condition on each strip $|t| \leq a$, $|x| < \infty$ for any $a > 0$. Show that the initial value problem

$$x' = f(t, x), \quad x(0) = x_0$$

has a solution which exists for all t. Note that a solution of the initial value problem $x' = x^2$ with $x(0) = -1$ exists only for $t > -1$. Does it violate the above statement?