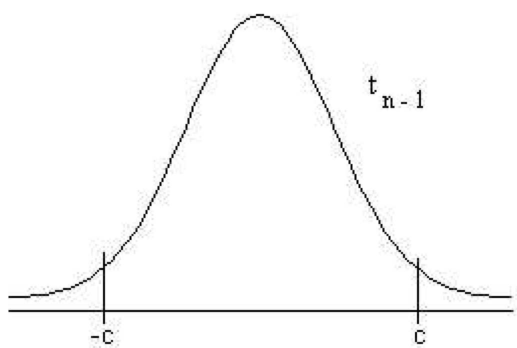
18.05 Lecture 31
April 29, 2005

t-test
$X_1, ..., X_n$ - a random sample from $N(\mu, \sigma^2)$

2-sided Hypothesis Test:
$H_1 : \mu = \mu_0$
$H_2 : \mu \neq \mu_0$
2 sided hypothesis - parameter can be greater or less than μ_0
Take $\alpha \in (0, 1)$ - level of significance (error of type 1)
Construct a confidence interval $\rightarrow$ confidence = 1 - α
If μ_0 falls in the interval, choose H_1, otherwise choose H_2

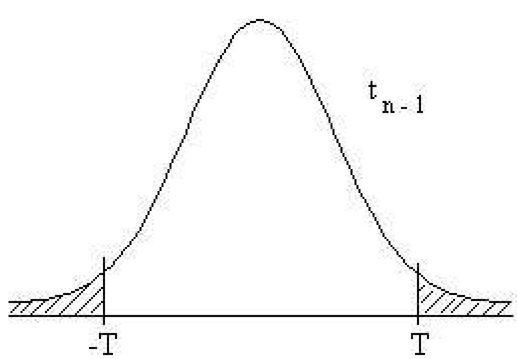
How to construct the confidence interval in terms of the decision rule:

$$\mathcal{T} = \frac{\overline{x} - \mu_0}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}} \sim \text{ t distribution with n - 1 degrees of freedom.}$$



Under the hypothesis $H_1, \mathcal{T}$ is has a t-distribution.
See if the $\mathcal{T}$ value falls in the expected area of the t-distribution:
Accept the null hypothesis (H_1), if $-c \leq T \leq c$, Reject if otherwise.
Choose c such that area between c and -c is $1 - \alpha$, each tail area = $\alpha/2$
Error of type 1: $\alpha_1 = \mathbb{P}_1(T < -c, T > c) = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$

Definition: p-value

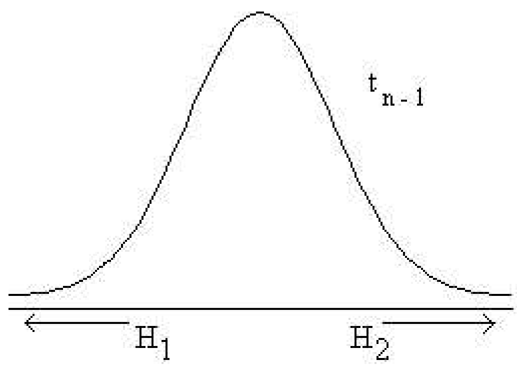


p-value = probability of values less likely than $\mathcal{T}$
If p-value $\geq \alpha$, accept the null hypothesis.
If p-value $< \alpha$, reject the null hypothesis.
Example: p-value = 0.0001, very unlikely that this $\mathcal{T}$ value would occur if the mean were μ_0. Reject the null hypothesis!

1-sided Hypothesis Test:
$H_1 : \mu \leq \mu_0$
$H_2 : \mu > \mu_0$

$$\mathcal{T} = \frac{\overline{x} - \mu_0}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}}$$

See how the distribution behaves for three cases:
1) If $\mu = \mu_0, \mathcal{T} \sim t_{n-1}$.



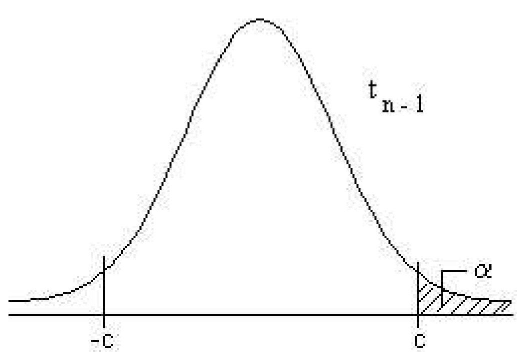
2) If $\mu < \mu_0$:

$$\mathcal{T} = \frac{\overline{x} - \mu_0}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}} = \frac{\mu - \mu_0}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}} + \frac{\overline{x} - \mu}{\sqrt{\frac{1}{n-1}(\overline{x^2} - (\overline{x})^2)}}$$

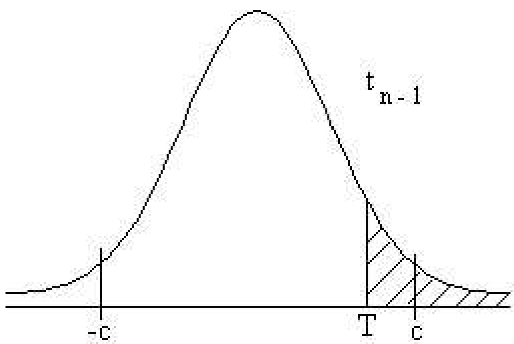
$$\mathcal{T} \approx t_{n-1} + \frac{(\mu - \mu_0)\sqrt{n-1}}{\sigma} \to -\infty$$

3) If $\mu > \mu_0$, similarly $\mathcal{T} \to +\infty$

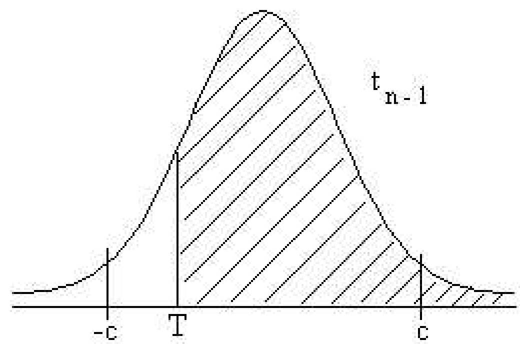
Decision Rule:
$\delta = \{H_1 : \mathcal{T} \leq \rfloor; \mathcal{H}_\in : \mathcal{T} > \rfloor\}$



$\alpha_1 = \mathbb{P}_1(T > c) = \alpha$
p-value: Still the probability of values less likely than $\mathcal{T}$,
but since it is 1-sided,
you don't need to consider the area to the left of $-\mathcal{T}$ as you would in the 2-sided case.

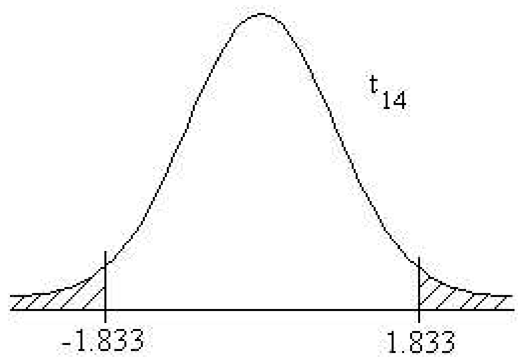


or

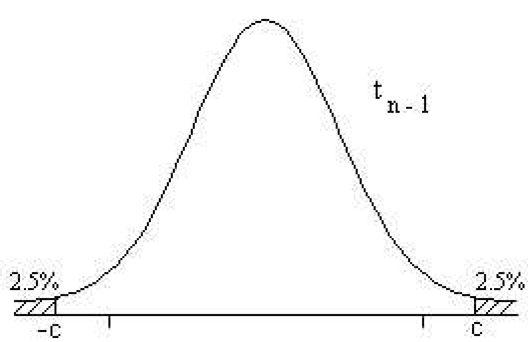


The p-value is the area of everything to the right of $\mathcal{T}$

Example: 8.5.1, 8.5.4
$\mu_0 = 5.2 . n = 15, \overline{x} = 5.4, \sigma' = 0.4226$
$H_1 : \mu = 5.2, H_2 : \mu \neq 5.2$
$\mathcal{T}$ is calculated to be $= 1.833$, which leads to a p-value of 0.0882



If $\alpha = 0.05$, accept H_1, $\mu = 5.2$. because the p-value is over 0.05
Decision rule:
Such that $\alpha = 0.05$, the areas of each tail in the 2-sided case = 2.5%



From the table → c = 2.145
$\delta = \{H_1 : -2.145 \leq T \leq 2.145; H_2 \text{ otherwise}\}$

Consider 2 samples, want to compare their means:
$X_1, ..., X_n \sim N(\mu_1, \sigma_1^2)$ and $Y_1, ..., Y_m \sim N(\mu_2, \sigma_2^2)$

Paired t-test:
Example (textbook): Crash test dummies, driver and passenger seats ∼ (X, Y)
See if there is a difference in severity of head injuries depending on the seat:
$(X_1, Y_1), ..., (X_n, Y_n)$
Observe the paired observations (each car) and calculate the difference:
Hypothesis Test:
$H_1 : \mu_1 = \mu_2$
$H_2 : \mu_1 \neq \mu_2$
Consider $Z_1 = X_1 - Y_1, ..., Z_n = X_n - Y_n \sim N(\mu_1 - \mu_2 = \mu, \sigma^2)$
$H_1 : \mu = 0; H_2 : \mu \neq 0$
Just a regular t-test:
p-values comes out as $< 10^{-6}$, so they are likely to be different.

** End of Lecture 31