18.05 Lecture 12
March 2, 2005

Functions of Random Variables

X - random variable, continuous with p.d.f. f(x)
Y = r(X)
Y doesn't have to be continuous, if it is, find the p.d.f.
To find the p.d.f., first find the c.d.f.

$$\mathbb{P}(Y \le y) = \mathbb{P}(r(X) \le y) = \mathbb{P}(x : r(X) \le y) = \int_{x:r(x)\le y} f(x)dx.$$

Then, differentiate the c.d.f to find the p.d.f.:

$$f(y) = \frac{\partial(\mathbb{P}(Y \le y))}{\partial y}$$

Example:
Take random variable X, uniform on [-1, 1]. Y = X^2, find distribution of Y.
p.d.f. $f(x) = \{\frac{1}{2}$ for $-1 \le x \le 1; 0$ otherwise$\}$

$$Y = X^2, \mathbb{P}(Y \le y) = \mathbb{P}(X^2 \le Y) = \mathbb{P}(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx$$

.

Take derivative before integrating.

$$\frac{\partial}{\partial y}\mathbb{P}(Y \le y) = f(\sqrt{y}) \times \frac{1}{2\sqrt{y}} + f(-y) \times \frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{y}}(f(\sqrt{y}) + f(-\sqrt{y}))$$

$$f(y) = \{\frac{1}{\sqrt{y}}, 0 \le y \le 1; 0 \text{ otherwise.}\}$$

Suppose r is monotonic (strictly one-to-one function).
$X = r(y)$, can always find inverse: $y = r^{-1}(x) = s(y)$ - inverse of r.
$\mathbb{P}(Y \le y) = \mathbb{P}(r(x) \le y) =$
$= \mathbb{P}(X \le s(y))$ if r is increasing (1)
$= \mathbb{P}(X \ge s(y))$ if r is decreasing (2)

(1) $= F(s(y))$ where $F()$ - c.d.f. of X,

$$\frac{\partial \mathbb{P}(Y \le y)}{\partial y} = \frac{\partial F(s(y))}{\partial y} = f(s(y))s'(y)$$

(2) $= 1 - \mathbb{P}(X < s(y)) = 1 - F(s(y))$,

$$-\frac{\partial \mathbb{P}(Y \le y)}{\partial y} = -f(s(y))s'(y)$$

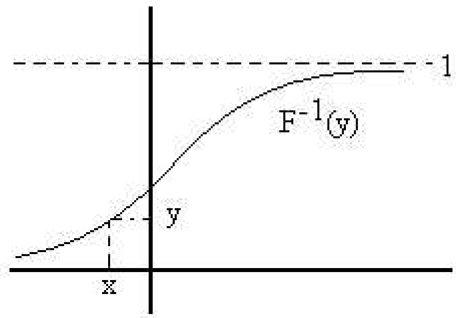
If r is increasing $\rightarrow s = r^{-1}$ is increasing. $\rightarrow s'(y) \ge 0 \rightarrow s'(y) = |s'(y)|$
If r is decreasing $\rightarrow s = r^{-1}$ is decreasing. $\rightarrow s'(y) \le 0 \rightarrow -s'(y) = |s'(y)|$
Answer: p.d.f. of Y: $f(y) = f(s(y))|s'(y)|$

Example: $f(x) = \{3(1-x)^2, 0 \le x \le 1; 0 \text{ otherwise.}\}$ $Y = 10e^{5x}$

$$Y = 10e^{5x} \to X = \frac{1}{5}\ln(\frac{Y}{10}); X` = \frac{1}{5Y}$$

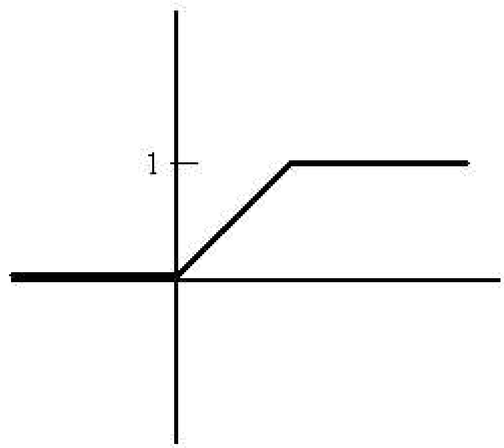
$$f(y) = 3(1 - \frac{1}{5}\ln(\frac{Y}{10})|\frac{1}{5Y}|, 10 \le y \le 10e^5; 0, \text{ otherwise.}$$

X has c.d.f. $F(X) = \mathbb{P}(X \le x)$, continuous.
$Y = F(X), 0 \le Y \le 1$, what is the distribution of Y?

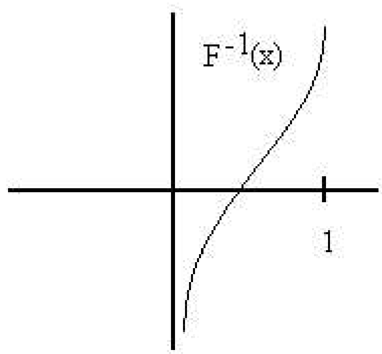


c.d.f. $\mathbb{P}(Y \le y) = \mathbb{P}(F(X) \le y) = \mathbb{P}(X \le F^{-1}(y)) = F(F^{-1}(y)) = y, 0 \le y \le 1$
p.d.f. $f(y) = \{1, 0 \le y \le 1; 0, \text{otherwise.}\}$

Y - uniform on interval [0, 1]



X - uniform on interval $[0, 1]$; F - c.d.f. of Y.



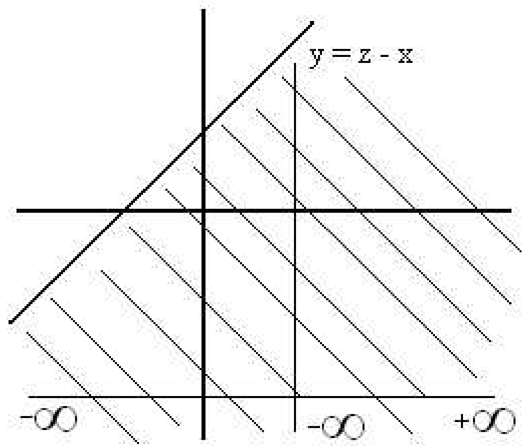
$Y = F^{-1}(X); \mathbb{P}(Y \leq y) = \mathbb{P}(F^{-1}(x) \leq y) = \mathbb{P}(X \leq F(y)) = F(y)$.
$\rightarrow$ Random Variable $Y = F^{-1}(X)$ has c.d.f. $F(y)$.

Suppose that (X, Y) has joint p.d.f. f(x, y). Z = X + Y.

$$\mathbb{P}(Z \leq z) = \mathbb{P}(X + Y \leq z) = \int_{x+y\leq z} f(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)dydx,$$

p.d.f.:

$$f(z) = \frac{\partial \mathbb{P}(Z \leq z)}{\partial z} = \int_{-\infty}^{\infty} f(x, z-x)dx$$



If X, Y independent, $f_1(x)$ = p.d.f. of X. $f_2(y)$ = p.d.f. of Y
Joint p.d.f.:

$$f(x,y) = f_1(x)f_2(y);\ f(z) = \int_{-\infty}^{\infty} f_1(x)f_2(z-x)dx$$

Example: X, Y independent, have p.d.f.:

$$f(x) = \{\alpha e^{-\alpha x}, x \geq 0; 0,\ \text{otherwise}\}.$$

$Z = X + Y$:

$$f(z) = \int_0^z \alpha e^{-\alpha x}\alpha e^{-\alpha(z-x)}dx$$

Limits determined by: $(0 \leq x, z - x \geq 0 \rightarrow 0 \leq x \leq z)$

$$f(z) = \alpha^2 \int_0^z e^{-\alpha z}dx = \alpha^2 e^{-\alpha z}\int_0^z dx = \alpha^2 z e^{-\alpha z}$$

This distribution describes the lifespan of a high quality product.
It should work “like new” after a point, given it doesn’t break early on.
Distribution of X itself:

$$X, \mathbb{P}(X \geq x) = \int_x^\infty \alpha e^{-\alpha x} dx = \alpha e^{-\alpha x}(-\frac{1}{\alpha})|_x^\infty = e^{-\alpha x}$$

Conditional Probability:

$$\mathbb{P}(X \geq x+t | X \geq x) = \frac{\mathbb{P}(X \geq x+t, x \geq x)}{\mathbb{P}(X \geq x)} = \frac{\mathbb{P}(X \geq x+t)}{\mathbb{P}(X \geq x)} = \frac{e^{-\alpha(X+t)}}{e^{-\alpha x}} = e^{-\alpha t} = \mathbb{P}(X \geq t)$$

** End of Lecture 12