18.05 Lecture 11
February 28, 2005

A pair (X, Y) of random variables:
f(x, y) joint p.f. (discrete), joint p.d.f. (continuous)
Marginal Distributions: $f(x) = \sum_y f(x, y)$ - p.f. of X (discrete)
$f(x) = \int f(x, y)dy$ - p.d.f. of X (continuous)

Conditional Distributions

Discrete Case:

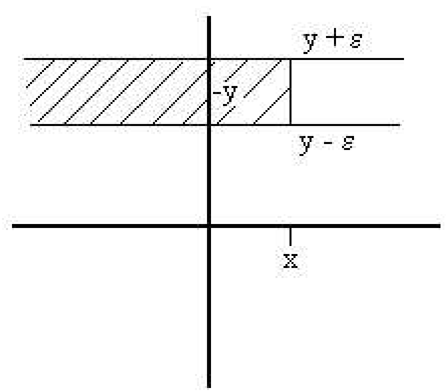
$$\mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

$\mathbb{P} = \frac{f(x,y)}{f(y)} = f(x|y)$ conditional p.f. of X given Y = y. Note: defined when f(y) is positive.
$f(y|x) = \frac{f(x,y)}{f(x)}$ conditional p.f. of Y given X = x. Note: defined when f(x) is positive.
If the marginal probabilities are zero, conditional probability is undefined.

Continuous Case:
Formulas are the same, but can't treat like exact possibilities at fixed points.
Consider instead in terms of probability density:
Conditional c.d.f. of X given Y=y;

$$\mathbb{P}(X \leq x|Y \in [y - \epsilon, y + \epsilon]) = \frac{\mathbb{P}(X \leq x, Y \in [y - \epsilon, y + \epsilon])}{\mathbb{P}(Y \in [y - \epsilon, y + \epsilon])}$$

Joint p.d.f. $f(x, y), \mathbb{P}(A) = \int_A f(x, y)dxdy$



$$= \frac{\frac{1}{2\epsilon}\int_{y-\epsilon}^{y+\epsilon}\int_{-\infty}^{x} f(x, y)dxdy}{\int_{y-\epsilon}^{y+\epsilon}\int_{-\infty}^{\infty} f(x, y)dxdy \times \frac{1}{2\epsilon}}$$

As $\epsilon \to 0$:

$$\frac{\int_{-\infty}^{x} f(x, y)dx}{\int_{-\infty}^{\infty} f(x, y)dx} = \frac{\int_{-\infty}^{x} f(x, y)dx}{f(y)}$$

Conditional c.d.f:

$$\mathbb{P}(X \leq x | Y = y) = \frac{\int_{-\infty}^{x} f(x,y)dx}{f(y)}$$

Conditional p.d.f:

$$f(x|y) = \frac{\partial}{\partial x}\mathbb{P}(X \leq x | Y = y) = \frac{f(x,y)}{f(y)}$$

Same result as discrete.
Also, $f(x|y)$ only defined when f(y) > 0.

Multiplication Rule

$f(x,y) = f(x|y)f(y)$
Bayes's Theorem:

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{f(x|y)f(y)}{\int f(x,y)dy} = \frac{f(x|y)f(y)}{\int f(x|y)f(y)dy}$$

Bayes's formula for Random Variables. For each y, you know the distribution of x.
Note: When considering the discrete case, $\int \rightarrow \sum$
In statistics, after observing data, figure out the parameter using Bayes's Formula.

Example: Draw X uniformly on [0, 1], Draw Y uniformly on [X, 1]
p.d.f.:

$$f(x) = 1 \times I(0 \leq x \leq 1), f(y|x) = \frac{1}{1-x} \times I(x \leq y \leq 1)$$

Joint p.d.f:

$$f(x,y) = f(y|x)f(x) = \frac{1}{1-x} \times I(0 \leq x \leq y \leq 1)$$

Marginal:

$$f(y) = \int f(x,y)dx = \int_0^y \frac{1}{1-x}dx = -\ln(1-x)|_0^y = -\ln(1-y)$$

Keep in mind, this condition is everywhere: given, $y \in [0,1]$ and f(y) = 0 if y $\notin$ [0, 1]
Conditional (of X given Y):

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{-1}{(1-x)\ln(1-y)} I(0 \leq x \leq y \leq 1)$$

Multivariate Distributions

Consider n random variables: $X_1, X_2, ..., X_n$
Joint p.f.: $f(x_1, x_2, ..., x_n) = \mathbb{P}(X_1 = x_1, ..., X_n = x_n) \geq 0, \sum f = 1$
Joint p.d.f.: $f(x_1, x_2, ..., x_n) \geq 0, \int f dx_1 dx_2 ... dx_n = 1$
Marginal, Conditional in the same way:
Define notation as vectors to simplify:
$\overrightarrow{X} = (X_1, ..., X_n), \overrightarrow{x} = (x_1, ..., x_n)$
$\overrightarrow{X} = (\overrightarrow{Y}, \overrightarrow{Z})$ subsets of coordinates: $\overrightarrow{Y} = (X_1, ..., X_k), \overrightarrow{y} = (y_1 ... y_k)$
$\overrightarrow{Z} = (X_{k+1}, ..., X_n), \overrightarrow{z} = (z_1 ... z_{n-k})$
Joint p.d.f. or joint p.f. of $\overrightarrow{X}, f(\overrightarrow{x}) = f(\overrightarrow{y}, \overrightarrow{z})$

Marginal:

$$f(\overrightarrow{y}) = \int f(\overrightarrow{y}, \overrightarrow{z})d\overrightarrow{z}, f(\overrightarrow{z}) = \int f(\overrightarrow{y}, \overrightarrow{z})d\overrightarrow{y}$$

Conditional:

$$f(\overrightarrow{y}|\overrightarrow{z}) = \frac{f(\overrightarrow{y}, \overrightarrow{z})}{f(\overrightarrow{z})}, f(\overrightarrow{z}|\overrightarrow{y}) = \frac{f(\overrightarrow{y}|\overrightarrow{z})f(\overrightarrow{z})}{\int f(\overrightarrow{y}|\overrightarrow{z})f(\overrightarrow{z})d\overrightarrow{z}}$$

Functions of Random Variables

Consider random variable X and a function r: $\mathbb{R} \to \mathbb{R}$,
$Y = r(X)$, and you want to calculate the distribution of Y.

Discrete Case:
Discrete p.f.:

$$f(y) = \mathbb{P}(Y = y) = \mathbb{P}(r(X) = y) = \mathbb{P}(x : r(x) = y) = \sum_{x:r(x)=y} \mathbb{P}(X = x) = \sum_{x:r(x)=y} f(x)$$

(very similar to "change of variable")

Continuous Case:
Find the c.d.f. of Y = r(X) first.

$$\mathbb{P}(Y \leq y) = \mathbb{P}(r(X) \leq y) = \mathbb{P}(x : r(x) \leq y) = \mathbb{P}(A(y)) = \int_{A(y)} f(x)dx$$

p.d.f. $f(y) = \frac{\partial}{\partial y} \int_{A(y)} f(x)dx$

** End of Lecture 11