

A Basis for 3 by 3 Symmetric Matrices

The real 3 by 3 matrices form a vector space M . The symmetric matrices in M form a subspace S . If you add two symmetric matrices, or multiply by real numbers, the result is still a symmetric matrix. **Problem: Find a basis for S .**

When I asked this question on an exam, I realized that a key point needs to be emphasized: **The basis “vectors” for S must lie in the subspace.** They are 3 by 3 symmetric matrices! Then there are two requirements:

1. The basis vectors must be linearly independent.
2. Their combinations must produce every vector (matrix) in S .

Here is one possible basis (all symmetric) for this example:

$$\begin{array}{lll} S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ S_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Since this basis contains 6 vectors, the **dimension of S is 6.**

Question: Find a basis for the subspace AS of 3 by 3 antisymmetric matrices (with $A^T = -A$). What is its dimension?

Bases for S and AS together give a basis for the whole space M (all 3 by 3 matrices). Write the upper triangular all-ones matrix U as a symmetric matrix plus an antisymmetric matrix.