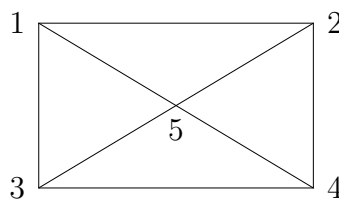


Your PRINTED name is: _____

Grading 1
2
3

- 1) (40 pts.) This problem is based on a 5-node graph.



I have not included edge numbers and arrows. Add them if you want to: not needed.

- (a) Find $A^T A$ for this graph. A is the incidence matrix.
- (b) The sum of the eigenvalues of $A^T A$ is _____ .
The product of those eigenvalues is _____ .
- (c) What is $A^T A$ for a graph with **only one edge**? How can that small $A^T A$ be used in constructing $A^T A$ for a large graph?
- (d) Suppose I want to solve $Au = \text{ones}(8, 1) = b$ by least squares. What equation gives a best \hat{u} ? For the incidence matrix A , is there exactly one best \hat{u} solving that equation? (If your equation has more than one best \hat{u} , describe the difference between any two solutions.)

a

The matrix A^tA is the graph Laplacian, given by $D - W$ where D is the degree matrix and W is the adjacency matrix. Therefore it is a 5×5 matrix:

$$\begin{bmatrix} 3 & -1 & -1 & 0 & -1 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 3 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

b

The trace of A^tA is the sum of the diagonal entries, so 16. The determinant is the product of the eigenvalues. Since any graph Laplacian has a null-vector (namely any constant vector), one of the eigenvalues is 0, so the determinant is also 0.

c

The A^tA for a one edge graph is

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

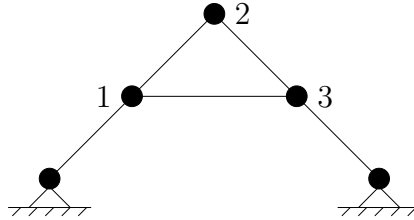
To construct A^tA for a larger graph from these *element* matrices, we construct one for each edge of the graph and add them together according to the following scheme. If there are n nodes, start with an $n \times n$ matrix of zeroes M . Then, for an edge connecting node i to node j , add 1 to M_{ii} and M_{jj} and -1 to M_{ij} and M_{ji} . Do this for each edge and the end result is A^tA for the larger graph.

d

The least squares approximation to $Au = b$ is given by the solution of $A^tA\hat{u} = A^tb$. In the case of the incidence matrix, there will not be one best solution, since for any solution \hat{u} minimizing square error, if v is in the nullspace of A , then $\hat{u} + v$ has the same square error.

Since A has non-trivial nullspace, there is no best solution. Any two solutions will differ by an element of the nullspace, namely a constant vector.

- 2) (30 pts.)
- (a) Suppose A is an m by n matrix of rank r (so it has r independent columns). How many independent solutions to $Au = 0$ and $A^T w = 0$?
- (b) Draw a full set of mechanisms (solutions to $e = Au = 0$ with no stretching) for this truss with unit length bars and 45° angles.



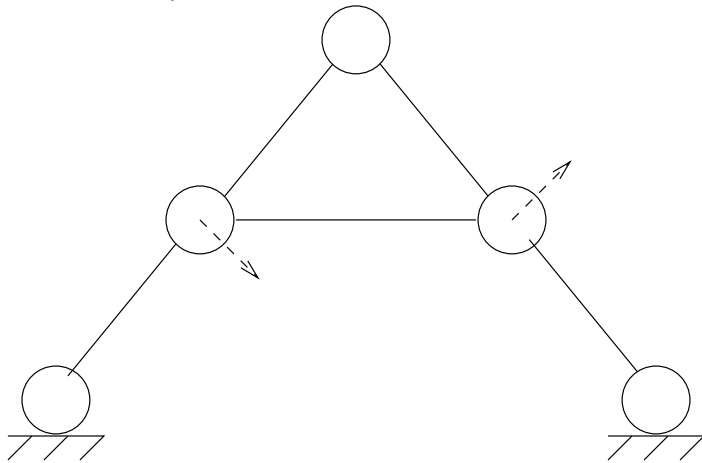
- (c) Suppose a mechanism has $u_1^H = .01$. What are u_1^V and u_3^H and u_3^V ?
What is the actual new length of the bar between joints 1 and 3?

a

An $m \times n$ matrix with rank r has $n - r$ independent solutions to $Au = 0$ and $m - r$ independent solutions to $A^t w = 0$. This is an application of the Rank-Nullity Theorem, and the fact that the rank of A^t equals the rank of A .

b

There is only one mechanism.



As a vector it is written

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

c

If node 1 moves .01 horizontally, then the corresponding mechanism is

$$\begin{bmatrix} .01 \\ -.01 \\ 0 \\ 0 \\ .01 \\ .01 \end{bmatrix}$$

so we have $u_1^H = u_3^H = u_3^V = .01$ and $u_1^V = -.01$. The actual new length of the bar connecting nodes 1 and 3 is

$$L_{new} = \sqrt{L_{old}^2 + (0.02)^2}$$

3) (30 pts.) This problem is about the equation

$$-u''(x) + u(x) = 1 \quad \text{with } u(0) = 0 \text{ and } u(1) = 0.$$

- (a) Multiply by a test function $v(x)$. Find the weak form of the equation, after an integration by parts.
- (b) With $h = \Delta x = \frac{1}{3}$ draw the admissible piecewise linear trial functions $\phi_1(x), \dots, \phi_n(x)$. What is n ? With test functions = trial functions, give a *formula* for the entry K_{12} in the finite element equation $KU = F$.
- (c) Find all the numbers in K and F .

a

First, multiply the equation by the test function, $v(x)$.

$$-u''(x)v(x) + u(x)v(x) = v(x)$$

Integrate from $x = 0$ to $x = 1$.

$$-\int_0^1 u''(x)v(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 v(x)dx$$

The first term can be integrated by parts like we did for the equation in the book.

$$\int_0^1 u''(x)v(x)dx = [u'(x)v(x)]_0^1 - \int_0^1 u'(x)v'(x)dx$$

Here's the first term (the integrated term) written out.

$$[u'(x)v(x)]_0^1 = u'(1)v(1) - u'(0)v(0)$$

Since the boundary conditions don't require $u'(0)$ or $u'(1)$ to necessarily be zero, to get rid of the integrated term we must have $v(0) = 0$ and $v(1) = 0$. (This is a requirement for the test functions.)

Assuming all the test functions $v(x)$ have $v(0) = 0$ and $v(1) = 0$, the integrated term equals zero. Plugging in the other term from the integration by parts gives you the weak form of the equation.

$$\int_0^1 u'(x)v'(x)dx + \int_0^1 u(x)v(x)dx = \int_0^1 v(x)dx$$

For this equation, the weak form has an additional term, the $\int_0^1 u(x)v(x)dx$ term, than the equation discussed in the book.

b

The piecewise linear trial functions $\phi_i(x)$ are the standard "hat" functions described in the book. For this problem, as we did in the example in the book, we're using the same trial functions as test functions. As mentioned above, the test functions must be zero at $x = 0$ and $x = 1$, so there are only two admissible trial functions, $\phi_1(x)$ centered at $x = \frac{1}{3}$ and

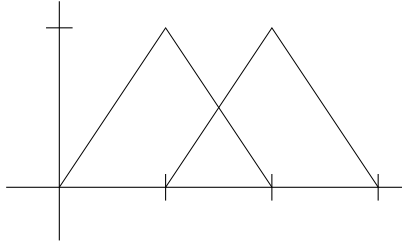


Figure 1: Admissible trial functions, centered at $x = \frac{1}{3}$ and $x = \frac{2}{3}$

$\phi_2(x)$ centered at $x = \frac{2}{3}$, both of which are zero at $x = 0$ and $x = 1$. So $n = 2$. See figure 1 for a crude drawing.

Because the weak form of the equation for this problem has an additional term than the equation in the book, the formula for K_{ij} has an additional term.

$$K_{ij} = \int_0^1 \phi_i'(x)\phi_j'(x)dx + \int_0^1 \phi_i(x)\phi_j(x)dx$$

Since the right hand side is the same as in the book, the formula for F_i is the same.

$$F_i = \int_0^1 \phi_i(x)dx$$

If you want to see where the new formula for K_{ij} comes from, substitute $u(x) = U_1\phi_1(x) + U_2\phi_2(x)$ and, in two separate equations, $v(x) = \phi_1(x)$ and $v(x) = \phi_2(x)$.

$$\begin{aligned} \int_0^1 (U_1\phi_1'(x) + U_2\phi_2'(x)) \phi_1'(x)dx + \int_0^1 (U_1\phi_1(x) + U_2\phi_2(x)) \phi_1(x)dx &= \\ & \int_0^1 \phi_1(x)dx \\ \int_0^1 (U_1\phi_1'(x) + U_2\phi_2'(x)) \phi_2'(x)dx + \int_0^1 (U_1\phi_1(x) + U_2\phi_2(x)) \phi_2(x)dx &= \\ & \int_0^1 \phi_2(x)dx \end{aligned}$$

To rewrite this as a linear system $KU = F$, regroup the terms to see what's multiplying U_1 and U_2 in each equation. (The constants U_1 and U_2 can be taken out of the integrals.) For

example, you can see what's multiplying U_2 in the first equation – this is K_{12} .

$$K_{12} = \int_0^1 \phi_2'(x)\phi_1'(x)dx + \int_0^1 \phi_2(x)\phi_1(x)dx$$

You can check that the general formula for K_{ij} above is right by looking at what multiplies U_j in equation i .

c

To actually calculate the entries of K and F , break up each integral into pieces, as appropriate. For example, take K_{11} .

$$\begin{aligned} K_{11} &= \int_0^1 \phi_1'(x)\phi_1'(x)dx + \int_0^1 \phi_1(x)\phi_1(x)dx \\ &= \int_0^{\frac{1}{3}} 3 \cdot 3dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3) \cdot (-3)dx \\ &\quad + \int_0^{\frac{1}{3}} (3x) \cdot (3x)dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (2-3x)(2-3x)dx \\ &= 3 + 3 + \frac{1}{9} + \frac{1}{9} \\ &= 6 + \frac{2}{9} \\ &= \frac{56}{9} \end{aligned}$$

Notice that, besides the 6 which appears in the K_{11} in the book, there's another term, $\frac{2}{9}$, because of the additional term in the weak form of the equation.

The result of calculating all the numbers K_{ij} is below.

$$K = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} + \begin{pmatrix} \frac{2}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{56}{9} & -\frac{53}{18} \\ -\frac{53}{18} & \frac{56}{9} \end{pmatrix}$$

(I may have made a mistake so feel free to let me know if you get different numbers!) The result of calculating F is the same as the first two entries of F in the book.

$$F = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$