

# 18.085 FALL 2002 QUIZ 3 SOLUTIONS

## PROBLEM 1

a) The graph looks like a symmetric butterfly. It is periodic with no discontinuities (but has kinks at  $x = 0$  and  $x = k\pi$ , where the slope jumps).

$$\begin{aligned} c_k &= \frac{1}{2\pi} \left( \int_{-\pi}^0 e^x e^{-ikx} dx + \int_0^\pi e^{-x} e^{-ikx} dx \right) \\ &= \frac{1}{2\pi} \left( \int_{-\pi}^0 e^{x-ikx} dx + \int_0^\pi e^{-x-ikx} dx \right) \\ &= \frac{1}{2\pi} \left( \frac{1}{1-ik} \left( 1 - e^{-\pi} (-1)^k \right) + \frac{1}{-1-ik} \left( e^{-\pi} (-1)^k - 1 \right) \right) \\ &= \frac{1 - e^{-\pi} (-1)^k}{\pi (k^2 + 1)}, \text{ using } e^{ik\pi} = e^{-ik\pi} = (-1)^k \end{aligned}$$

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b)  $df/dx$ : The right half of the graph becomes  $-e^{-x}$ : now it's an odd butterfly.  $d^2f/dx^2$ : Back to the even butterfly with  $\delta$ -functions  $-2\delta(x)$  and  $2e^{-\pi}\delta(x-\pi)$ . The  $\delta$ -functions come from jumps in  $df/dx$ .

c) From b)

$$-\frac{d^2 f}{dx^2} + f = 2\delta(x) - 2e^{-\pi}\delta(x-\pi)$$

If

$$f(x) = \sum c_k e^{ikx}$$

then (recall  $\delta(x) = \frac{1}{2\pi} \sum e^{ik\pi}$ ,  $\frac{d}{dx} \rightarrow ik$ ):

$$\begin{aligned} \sum c_k k^2 e^{ikx} + \sum c_k e^{ikx} &= \frac{1}{\pi} \sum e^{ikx} - \frac{1}{\pi} e^{-\pi} \sum e^{ik(x-\pi)} \\ &= \frac{1}{\pi} \sum e^{ikx} - \frac{1}{\pi} (-1)^k e^{-\pi} \sum e^{ikx} \end{aligned}$$

Equate coefficients of like terms:

$$(k^2 + 1) c_k = \frac{1}{\pi} \left( 1 - (-1)^k e^{-\pi} \right)$$

The solution of this algebraic equation is the same answer as before,

$$c_k = \frac{1 - (-1)^k e^{-\pi}}{\pi (k^2 + 1)}$$

## PROBLEM 2

a) The easiest way to perform this convolution is the matrix way:

$$u * v = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

b)

$$c = F^{-1}u = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$d = F^{-1}v = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

$$h = cd = \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{4} \\ 0 \end{bmatrix}$$

Then

$$u * v = 4Fh = 4 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

The answer in part a) is confirmed!

c) We have

$$DFT(u * z) = \left( \frac{1}{2}C_1, 0, \frac{1}{2}C_3, 0 \right)$$

Therefore, we need  $C_1 = 0$ ,  $C_3 = 0$  in order to have  $u * z = (0, 0, 0, 0)$ . So the Fourier coefficients of  $z$  must be  $(0, a, 0, b)$  for any  $a$  and  $b$ . Then

$$z = F \begin{bmatrix} 0 \\ a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} a + b \\ i(a - b) \\ -(a + b) \\ -i(a - b) \end{bmatrix}$$

This can be expressed more simply as

$$z = \begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix}$$

We can check this result by convolution:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ -x \\ -y \end{bmatrix} = 0$$

### PROBLEM 3

a) Since  $f(x)$  vanishes outside of  $[-1, 1]$  the limits of integration are  $-1$  and  $1$ :

$$\hat{f}(k) = -\int_{-1}^0 e^{-ikx} dx + \int_0^1 e^{-ikx} dx = 2i \frac{\cos k - 1}{k}$$

b) Multiply by  $ik$  for the derivative:

$$\hat{D}(k) = 2 - e^{ik} - e^{-ik} = -2(\cos k - 1)$$

Convolution becomes multiplication in the Fourier domain. Therefore the transform is

$$\hat{C}(k) = \hat{D}(k) \hat{f}(k) = -4i \frac{(\cos k - 1)^2}{k}.$$

This confirms that the derivative  $D = df/dx$  consists of  $\delta$ -functions at  $x = 0, -1, 1$ .