Solutions - Problem Set 4

Section 2.3

7) $b = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad x = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$

$$Au = b$$

Normal Equation

$$A^{\mathrm{T}} A\widehat{u} = A^{\mathrm{T}} b$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \widehat{u} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \widehat{u} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\widehat{u} = \frac{1}{20} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Nearest Line, $\widehat{C} + \widehat{D}x = 3 - x$ #

8) $p = A\widehat{u}$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$

For $y = 3 - x$

at $x = 0$,
$y = 3 - 0$
$= 3$

at $x = 1$,
$y = 2$

at $x = 2$,
$y = 1$

at $x = 3$,
$y = 0$

$\therefore$ Those four values do lie on the line $C + Dx$ #

$$
\begin{aligned}
l &= b - p \\
&= \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$A^{\mathrm{T}} e = 0$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ Verified that $A^{\mathrm{T}} e = 0$ #

12) Parabola $C + Dx + Ex^2$

$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}}^{A} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \overbrace{\begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}}^{b}$$

Normal Equation

$$A^{\mathrm{T}} A \widehat{u} = A^{\mathrm{T}} b$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 10 \end{pmatrix}$$

Using MATLAB,

$$\begin{pmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$$

Cubic $C + Dx + Ex^2 + Fx^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

If I fit the best cubic $C + Dx + Ex^2 + Fx^3$ to those four points,

$$Au = b$$

can be solved directly by

$$u = A^{-1}b$$

since A is invertible

$$\begin{aligned} e &= b - Au \\ &= b - A(A^{-1}b) \\ &= b - b \\ &= 0 \end{aligned}$$

The error vector $e = \mathbf{0}$ #

Proof:

By gaussian elimination,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 9 & 27 \end{pmatrix} \widehat{u} = \begin{pmatrix} 4 \\ -3 \\ -4 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 6 & 24 \end{pmatrix} \widehat{u} = \begin{pmatrix} 4 \\ -3 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \widehat{C} \\ \widehat{D} \\ \widehat{E} \\ \widehat{F} \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \widehat{F} &= 0 \\ \widehat{E} &= 1 \\ \widehat{D} &= -4 \\ \widehat{C} &= 4 \end{aligned}$$

$$\begin{aligned} e &= \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{\#} \end{aligned}$$

18) $\widehat{u}_9 = \frac{1}{9}(u_1 + u_2 + \cdots + u_9)$

$$\widehat{u}_{10} = \frac{1}{10}(u_1 + u_2 + \cdots + u_{10})$$

$$= \frac{1}{10}u_{10} + \frac{1}{10}(u_1 + u_2 + \cdots + u_9)$$

$$= \frac{1}{10}u_{10} + \frac{9}{10}\widehat{u}_9 \ _{\#}$$

22) $t = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}$$

Normal Equation

$$A^{\mathrm{T}}A\widehat{u} = A^{\mathrm{T}}b$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \widehat{u} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \widehat{u} = \begin{pmatrix} 35 \\ 42 \end{pmatrix}$$

$$\widehat{u} = \frac{1}{14} \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 35 \\ 42 \end{pmatrix}$$

$$= \begin{pmatrix} 9 \\ 4 \end{pmatrix}_{\#}$$

The closest line is $= 9 - 4x$ $_{\#}$

$$b = A\widehat{u} - b$$

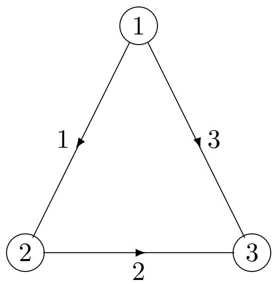
$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 9 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{\#}$$

The error $e = 0$ because this b is linear combinations of A $_{\#}$

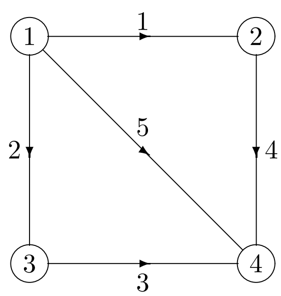
Section 2.4

1)



Incidence Matrix

$$\begin{array}{c} \\ A_{\textsf{triangle}} = \end{array} \begin{array}{c} \text{Node } 1 \quad 2 \quad 3 \\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{r} \\ \text{Edge } 1 \\ 2 \\ 3 \end{array} \quad \#$$



Incidence Matrix

$$\begin{array}{c} \\ A_{\textsf{square}} = \end{array} \begin{array}{c} \text{Node } 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{r} \\ \text{Edge } 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

$$A_{\textsf{triangle}}{}^{\mathrm{T}} A_{\textsf{triangle}} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \#$$

$$A_{\mathsf{square}}{}^{\mathrm{T}} A_{\mathsf{square}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}_{\#}$$

3) $A_{\mathsf{square}}\, u = 0$

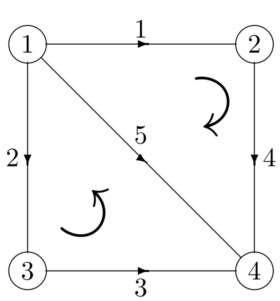
$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

one solution is the constant vector

$$u = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}_{\#}$$

$A_{\mathsf{square}}{}^{\mathrm{T}} w = 0$

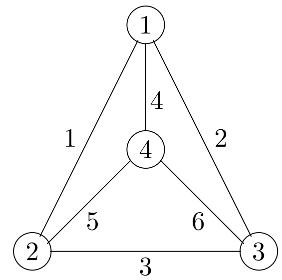
$$\begin{pmatrix} -1 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



2 solutions are:

$$W = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}_{\#}$$

8) Element matrix for edge i connecting node j and k



$$K_i = \begin{bmatrix} C_i & -C_i \\ -C_i & C_i \end{bmatrix} \begin{matrix} \text{row } j \\ \text{row } k \end{matrix}$$

$$K_1 = \begin{pmatrix} C_1 & -C_1 \\ -C_1 & C_1 \end{pmatrix} \quad K_2 = \begin{pmatrix} C_2 & -C_2 \\ -C_2 & C_2 \end{pmatrix} \quad K_3 = \begin{pmatrix} C_3 & -C_3 \\ -C_3 & C_4 \end{pmatrix}$$

$$K_4 = \begin{pmatrix} C_4 & -C_4 \\ -C_4 & C_4 \end{pmatrix} \quad K_5 = \begin{pmatrix} C_5 & -C_5 \\ -C_5 & C_5 \end{pmatrix} \quad K_6 = \begin{pmatrix} C_6 & -C_6 \\ -C_6 & C_6 \end{pmatrix}$$

$$A^{\mathrm{T}}CA = \begin{pmatrix} C_1 & -C_1 & 0 & 0 \\ -C_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} C_2 & 0 & -C_2 & 0 \\ 0 & 0 & 0 & 0 \\ -C_2 & 0 & C_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_3 & -C_3 & 0 \\ 0 & -C_3 & C_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} C_4 & 0 & 0 & -C_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C_4 & 0 & 0 & C_4 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_5 & 0 & -C_5 \\ 0 & 0 & 0 & 0 \\ 0 & -C_5 & 0 & C_5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & C_6 & -C_6 \\ 0 & 0 & -C_6 & C_6 \end{pmatrix}$$

$$= \begin{pmatrix} C_1+C_2+C_4 & -C_1 & -C_2 & -C_4 \\ -C_1 & C_1+C_3+C_5 & -C_3 & -C_5 \\ -C_2 & -C_3 & C_2+C_3+C_6 & -C_6 \\ -C_4 & -C_5 & -C_6 & C_4+C_5+C_6 \end{pmatrix}_{\#}$$

If all $C_i = 1$

$$A^{\mathrm{T}}A = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}_{\#}$$

12) $$K = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} \quad K^{-1} = \frac{1}{n}\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdots & & \cdots & \\ 1 & 1 & \cdots & 2 \end{pmatrix}$$

$$KK^{-1} = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} \frac{1}{n}\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdots & & \cdots & \cdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}$$

$$= \frac{1}{n}\begin{pmatrix} 2(n-1)-1(n-2) & (n-1)-2-(n-3) & \cdots \\ -2+(n-1)-(n-3) & -1+2(n-1)-(n-3) & \cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots \\ -2-(n-3)+(n-1) & -1-2-(n-4)+(n-1) & \cdots \end{pmatrix}$$

$$= \frac{1}{n}\begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} = I_{\#\ \text{verified}}$$

$$K = \underbrace{\begin{pmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \cdots & n-1 \end{pmatrix}}_{n-1} \Bigg\} n-1$$

$\det(n-1) = n-1$

$$\det\begin{pmatrix} n-1 & -1 \\ -1 & n-1 \end{pmatrix} = (n-1)^2 - 1$$
$$= n^2 - 2n$$
$$= n(n-2) > 0 \quad \text{for } n > 2$$

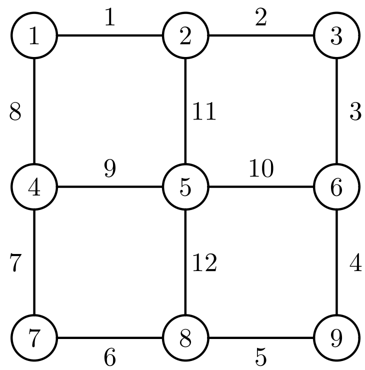
$$\det\begin{pmatrix} n-1 & -1 & -1 \\ -1 & n-1 & -1 \\ -1 & -1 & n-1 \end{pmatrix} > 0$$

the upper left determinants > 0
$\Rightarrow$ positive definite

Alternatively,
the eigenvalues of K are $\lambda = 1, n, n \ldots, n > 0$

Hence matrix K is positive definite$_{\#}$

17)



a) Among the 81 entries of $A^{T}A$ there are $9 + 12(2) = 33$ entries of non-zero values

$\therefore$ Zero entries in $A^{T}A = 81 - 33$

$= 48_{\#}$

b)
$$D = \begin{bmatrix} 2 & & & & & & & & \\ & 3 & & & & & & & \\ & & 2 & & & 0 & & & \\ & & & 3 & & & & & \\ & & & & 4 & & & & \\ & & & & & 3 & & & \\ & & & & & & 2 & & \\ & & 0 & & & & & 3 & \\ & & & & & & & & 2 \end{bmatrix}_{\#}$$

c) The middle row has $d_{55} = 4$ because node 5 has 4 edges connected to it.

There are four -1's in $-w$ because it is next to nodes $2, 4, 6$ and 8 $_{\#}$