18.085 Quiz 3 December 2, 2004 Professor Strang

Your name is: **SOLUTIONS**

Grading 1.
2.
3.

Thank you for taking 18.085 ! I hope to see you in 18.086 !!

1) **(40 pts.)** This question is about 2π-*periodic functions.*

(a) Suppose $f(x) = \sum c_k e^{ikx}$ and $g(x) = \sum d_l e^{ilx}$. Substitute for f and g and integrate to find the coefficients q_n in this convolution:

$$h(x) = \int_0^{2\pi} f(t)\, g(x-t)\, dt = \int_0^{2\pi} f(x-t)\, g(t)\, dt = \sum q_n e^{inx} .$$

(b) Compute the coefficients c_k for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{for } 1 \le x \le 2\pi \end{cases}$$

What is the *decay rate* of these c_k ? What is $\sum |c_k|^2$?

(c) Keep that $f(x)$ in parts (c)–(d). If $g(x)$ also has a jump, will the convolution $h(x)$ have a jump ? Compare the decay rates of the d's and q's to find the behavior of $h(x)$: delta function, jump, corner, or what ?

(d) Find the derivative dh/dx at $x = 0$ in terms of two values of $g(x)$. (You could take the x derivative in the convolution integral.)

Solution 1.

(a) This is really proving the convolution rule (periodic case).

$$\begin{aligned} h(x) &= \int_0^{2\pi} \left(\sum c_k e^{ikt}\right)\left(\sum d_l e^{il(x-t)}\right) dt \\ &= \int_0^{2\pi} \sum c_k d_k e^{ikx}\, dt \quad \text{(since all integrals of } e^{ikt}e^{-ilt} \text{ are zero if } k \neq l) \\ &= 2\pi \sum c_k d_k e^{ikx}\,. \end{aligned}$$

So $q_n = \boxed{2\pi c_n d_n}$ = multiplication in k-space. Note $c_n d_n$ (not $c_k d_l$).

(b) $c_k = \displaystyle\int_0^1 e^{-ikx}\, dx = \left.\frac{e^{-ikx}}{-ik}\right]_0^1 = \frac{1 - e^{-ik}}{ik}$ The decay rate is $1/k$.

By the energy identity $\displaystyle\sum |c_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\, dx = \frac{1}{2\pi}.$

(c) If $f(x)$ and $g(x)$ have jumps, their convolution $h = f * g$ has a *corner*. To see this in transform space from decay rates: c_k and d_k decay like $1/k$, so $c_k d_k$ decays like $1/k^2$.

(d) The derivative of the convolution integral (second form) gives

$$\frac{dh}{dx} = \int_0^{2\pi} \frac{df}{dx}(x-t)\, g(t)\, dt = \int_0^{2\pi} [\delta(x-t) - \delta(x-t-1)]\, g(t)\, dt$$

At $x = 0$ this is $\displaystyle\int_0^{2\pi} \delta(-t)\, g(t)\, dt - \int_0^{2\pi} \delta(-t-1)\, g(t)\, dt$

The spikes are at $t = 0$ and $t = -1$: $= \boxed{g(0) - g(-1)}$.

Or you could take the derivative of the form $h(x) = \displaystyle\int_0^{2\pi} f(t)\, g(x-t)\, dt$ to get

$$\int_0^{2\pi} f(t)\, g'(x-t)\, dt = \int_0^1 g'(x-t)\, dt = -g(x-t)\Big]_{t=0}^1 = -g(x-1) + g(x)$$

At $x = 0$ this is $\boxed{-g(-1) + g(0)}$.

The general rule is $h'(x) = f'(x) * g(x) = f(x) * g'(x)$. But NOT $h'(x) = f'(x) * g'(x)$. In k-space we have $q_k = 2\pi\, c_k d_k$ so for the derivative $ikq_k = 2\pi(ikc_k)d_k = 2\pi c_k(ikd_k)$. But NOT $ikq_k = 2\pi(ikc_k)(ikd_k)$.

2) **(30 pts.)**

(a) We want to compute the *cyclic convolution* of $f = (1, 0, 1, 0)$ and $g = (1, 0, -1, 0)$ in two ways. First compute $f *_C g$ directly—either the formula at the end of p. 294 or from $1 + w^2$ and $1 - w^2$.

(b) Now compute the discrete transforms c (from f) and d (from g). Then use the convolution rule to find $f *_C g$.

(c) I notice that the usual dot product $\overline{f}^{\mathrm{T}} g$ is zero. Maybe also $\overline{c}^{\mathrm{T}} d$ is zero. Question for any c and d:

$$\text{If} \quad \overline{c}^{\mathrm{T}} d = 0 \quad \text{deduce that} \quad \overline{f}^{\mathrm{T}} g = 0.$$

Solution 2.

(a) Directly $(1,0,1,0) *_C (1,0,-1,0) = \boxed{(0,0,0,0)}$. From $1+w^2$ times $1-w^2$ we get $1-w^4=0$ because $w^4=0$. From the circulant matrix we again get

$$f *_C g = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Columns 0 and 2 of F^{-1} are $\frac{1}{4}(1,1,1,1)$ and $\frac{1}{4}(1,-1,1,-1)$. Then $c = F^{-1}f = \frac{1}{4}(2,0,2,0)$ and $d = F^{-1}g = \frac{1}{4}(0,2,0,2)$. So every $c_k d_k = 0$. Transforming back gives $\boxed{f *_C g = \text{zero vector}}$.

(c) Suppose $\overline{c}^{\mathrm{T}} d = 0$. Then $\overline{f}^{\mathrm{T}} g = (\overline{F}\overline{c})^{\mathrm{T}}(Fd) = \overline{c}^{\mathrm{T}}\overline{F}^{\mathrm{T}} F d = n\overline{c}^{\mathrm{T}} d = 0$. QED.

Note: This is different from $f *_C g = 0$! That would not follow from $\overline{c}^{\mathrm{T}} d = 0$, you need every $c_k d_k = 0$.

3) **(30 pts.)** This question uses the Fourier integral to study

$$-\frac{d^2u}{dx^2} + u(x) = \begin{cases} 1 & \text{for } -1 \le x \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

(a) Take Fourier transforms of both sides to find a formula for $\widehat{u}(k)$.

(b) What is the decay rate of this $\widehat{u}$? At what points x is the solution $u(x)$ not totally smooth? Describe $u(x)$ at those points: delta, jump in $u(x)$, jump in du/dx, jump in d^2u/dx^2, or what?

(c) We know that the Green's function for this equation (when the right side is $\delta(x)$) is

$$G(x) = \tfrac{1}{2}e^{-|x|} = \begin{cases} \tfrac{1}{2}e^{-x} & \text{for } x \ge 0 \\ \tfrac{1}{2}e^{x} & \text{for } x \le 0 \end{cases}$$

Find the solution $u(x)$ at the particular point $x = 2$.

Solution 3.

(a) $(k^2+1)\,\widehat{u}(k) = \displaystyle\int_{-1}^{1} e^{-ikx}\,dx = \frac{e^{-ikx}}{-ik}\Big]_{-1}^{1} = \frac{e^{ik}-e^{-ik}}{ik} = \frac{2\sin k}{k}.$

Then $\widehat{u}(k) = \dfrac{2\sin k}{(k^2+1)k}.$

(b) Decay rate is $1/k^3$. So $u(x)$ has a $\boxed{\text{jump in } d^2u/dx^2}$ at the points $x=-1$ and $x=1$. You can see that from the differential equation: since $u(x)$ is continuous, the jumps in $f(x)$ on the right side must come from jumps in $u''(x)$ on the left side.

(c) The solution $u = G * f$ at the point $x = 2$ is the integral of responses at $x = 2$ to right sides $f = 1$ over $-1 \le x \le 1$. The distance t to $x = 2$ ranges from 3 to 1. Green's response function $\frac{1}{2}e^{-x}$ ranges from $\frac{1}{2}e{-3}$ to $\frac{1}{2}e^{-1}$:

$$u(2) = \frac{1}{2}\int_1^3 e^{-t}\,dt = \boxed{\frac{e^{-1}-e^{-3}}{2}}.$$

This comes directly from $G * f$ at $x = 2$. The word "responses" is used to help explain why this becomes an integral of $G(x)$ from $x = 1$ to $x = 3$.