

Your PRINTED name is: SOLUTIONSGrading 1  
2  
3  

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- 1) (36 pts.) (a) Suppose  $u(x)$  is linear on each side of  $x = 0$ , with slopes  $u'(x) = A$  on the left and  $u'(x) = B$  on the right:

$$u(x) = \begin{cases} Ax & \text{for } x \leq 0 \\ Bx & \text{for } x \geq 0 \end{cases}$$

What is the second derivative  $u''(x)$ ? Give the answer at every  $x$ .

- (b) Take discrete values  $U_n$  at all the whole numbers  $x = n$ :

$$U_n = \begin{cases} An & \text{for } n \leq 0 \\ Bn & \text{for } n \geq 0 \end{cases}$$

For each  $n$ , what is the second difference  $\Delta^2 U_n$ ? Using coefficients  $1, -2, 1$  (notice signs!) give the answer  $\Delta^2 U_n$  at every  $n$ .

- (c) Solve the differential equation  $-u''(x) = \delta(x)$  from  $x = -2$  to  $x = 3$  with boundary values  $u(-2) = 0$  and  $u(3) = 0$ .
- (d) Approximate problem (c) by a difference equation with  $h = \Delta x = 1$ . What is the matrix in the equation  $KU = F$ ? What is the solution  $U$ ?

*Solutions.*

(a)  $u''(x) = (B - A)\delta(x)$ . This is not  $B - A$  at  $x = 0$ .

$$(b) \Delta^2 u_n = \begin{cases} B - A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$(c) u(x) = \begin{cases} 3/5(x + 2) & -2 \leq x \leq 0 \\ 2/5(3 - x) & 0 \leq x \leq 3 \end{cases}$$

$$(d) K_{+++} = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \quad U = \begin{bmatrix} 3/5 \\ 6/5 \\ 4/5 \\ 2/5 \end{bmatrix} \quad \begin{array}{l} \text{fixed-} \\ \text{fixed} \end{array}$$

$U$  lies right on the graph of  $u(x)$

2) (24 pts.) A symmetric matrix  $K$  is “*positive definite*” if  $u^T K u > 0$  for every nonzero vector  $u$ .

(a) Suppose  $K$  is positive definite and  $u$  is a (nonzero) eigenvector, so  $Ku = \lambda u$ . From the definition above *show that*  $\lambda > 0$ . What solution  $u(t)$  to  $\frac{du}{dt} = Ku$  comes from knowing this eigenvector and eigenvalue?

(b) Our second-difference matrix  $K_4$  has the form  $A^T A$ :

$$K_4 = \begin{bmatrix} 1 & -1 & & & & & \\ & 1 & -1 & & & & \\ & & 1 & -1 & & & \\ & & & 1 & -1 & & \\ & & & & 1 & -1 & \\ & & & & & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & \\ -1 & 1 & & & & & \\ & -1 & 1 & & & & \\ & & -1 & 1 & & & \\ & & & -1 & 1 & & \\ & & & & -1 & 1 & \\ & & & & & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \\ & & & & & -1 & 2 \end{bmatrix}$$

Convince me how  $K_4 = A^T A$  proves that  $u^T K_4 u = u^T A^T A u > 0$  for every nonzero vector  $u$ . (Show me why  $u^T A^T A u \geq 0$  and why  $> 0$ .)

(c) This matrix is positive definite for which  $b$ ? Semidefinite for which  $b$ ? What are its pivots??

$$S = \begin{bmatrix} 2 & b \\ b & 4 \end{bmatrix}$$

*Solutions.* (8 pts each)

(a) "... show that  $\lambda > 0$ "

$$\begin{aligned} Ku &= \lambda u & \text{so } \lambda &= \frac{u^T Ku}{u^T u} > 0 \\ u^T Ku &= \lambda u^T u \end{aligned}$$

"What solution  $u(t)$  ..."

$$u(t) = e^{\lambda t} u$$

(b)  $u^T A^T A u = (Au)^T (Au) \geq 0$

The particular matrix  $A$  in this problem has independent columns. The only solution to  $Au = 0$  is  $u = 0$ . So  $u^T A^T A u > 0$  except when  $u = 0$ . Other proofs are *possible*.

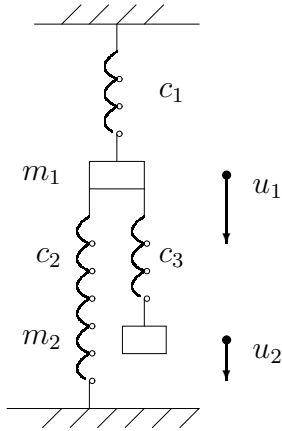
(c) Positive definite if  $b^2 < 8$

Semidefinite if  $b^2 = 8$

Pivots 2 and  $4 - \frac{b^2}{2}$

3) (40 pts.)

(a) Suppose I measure (with possible error)  $u_1 = b_1$  and  $u_2 - u_1 = b_2$  and  $u_3 - u_2 = b_3$  and finally  $u_3 = b_4$ . What matrix equation would I solve to find the best least squares estimate  $\hat{u}_1, \hat{u}_2, \hat{u}_3$ ? Tell me the *matrix* and the *right side* in  $K \hat{u} = f$ .



(b) What 3 by 2 matrix  $A$  gives the spring stretching  $e = Au$  from the displacements  $u_1, u_2$  of the masses?

(c) Find the stiffness matrix  $K = A^T C A$ . Assuming positive  $c_1, c_2, c_3$  show that  $K$  is invertible and positive definite.

Solutions.

(a) (12 points)

$$\begin{array}{l}
 Au = b \quad \text{is} \\
 \\
 \underbrace{A^T A}_K \hat{u} = \underbrace{A^T b}_f \quad \text{is} \\
 \underbrace{\begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & 1 \end{bmatrix}}_K \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \\
 \underbrace{\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix}}_K \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} b_2 - b_1 \\ b_3 - b_2 \\ b_4 + b_3 \end{bmatrix}}_f
 \end{array}$$

$\underbrace{\hspace{10em}}_{3pts \text{ for numbers}} \qquad \underbrace{\hspace{10em}}_{3pts}$

(b) (10 points)  $A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}$

(c) (16 points)  $A^T C A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + c_3 & -c_3 \\ -c_3 & c_3 \end{bmatrix}$

Any of these proofs is OK:

- (1)  $\det = c_1 c_3 + c_2 c_3 > 0$
- (2)  $A^T C A$  always positive definite with independent columns in  $A$
- (3) other ideas ...